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THE Σ-INVARIANTS OF S-ARITHMETIC SUBGROUPS OF BOREL GROUPS

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Dissertation zur Erlangung des
Doktorgrades der Mathematik (Dr. math.)

vorgelegt bei der Fakultät für Mathematik
der Universität Bielefeld

von EDUARD SCHESLER
aus Medwedizkij

betreut durch
PD DR. STEFAN WITZEL

Bielefeld,
im Januar 2020

Eduard Schesler: *The Σ-invariants of S-arithmetic subgroups of Borel groups*

Für Cynthia

ABSTRACT

Given a Chevalley group $\mathcal{G}$ of classical type and a Borel subgroup $\mathcal{B} \subset \mathcal{G}$ we compute the full Σ-invariants of the S-arithmetic groups $\mathcal{B}_d(\mathbb{Z}[1/p])$ for all but finitely many primes p. We introduce a class of height function X^* on Euclidean buildings X that naturally extends the class of Busemann functions on X. If X is thick enough we are able to determine the essential connectivity properties of the systems of superlevelsets in X that correspond to the height functions in X^*. In order to do so we develop a method that allows us to pull cycles that sit at the boundary at infinity of a Euclidean building into the building itself. Furthermore we will introduce new techniques in combinatorial Morse theory which, for the first time in the study of Σ-invariants, enable us to take advantage of the concept of essential n-connectivity rather than just n-connectivity.

ZUSAMMENFASSUNG

Sei $\mathcal{G}$ eine Chevalleygruppe vom klassischen Typ und sei $\mathcal{B} \subset \mathcal{G}$ eine borelsche Untergruppe. Wir bestimmen die vollständigen Σ-Invarianten der S-arithmetischen Gruppen $\mathcal{B}_d(\mathbb{Z}[1/p])$ für alle bis auf endlich viele Primzahlen p. Wir führen eine Klasse von Höhenfunktionen auf euklidischen Gebäuden X ein welche die Klasse der Busemannfunktionen auf X in natürlicher Weise erweitert. Unter der Voraussetzung, dass X dick genug ist bestimmen wir die essentiellen Zusammenhangseigenschaften der Systeme von Superlevelmengen in X welche durch die Höhenfunktionen in X^* induziert werden. Um dies zu bewerkstelligen entwickeln wir eine Methode die es uns erlaubt Zykel, die sich im unendlichen fernen Rand eines euklidischen Gebäudes befinden, in das Gebäude selbst hineinzuziehen. Desweiteren werden wir neue Techniken im Bereich der kombinatorischen Morse Theorie entwickeln die es uns, zum ersten mal bei Berechnungen von Σ-Invarianten, ermöglichen den Vorteil aus dem Konzept des essentiellen n-Zusammenhangs gegenüber des herkömmlichen n-Zusammenhangs herauszustellen.

ACKNOWLEDGMENTS

First and foremost, I would like to thank my advisor Stefan Witzel for his positive encouragement, understanding, and for his endless patience. I am grateful to him for always having an eye of the progress in my thesis and other parts of my scientific career.

Moreover, I would like to thank Kevin Wortman, not only for agreeing to act as a referee for this thesis, but also for the warm hospitality during my stay in Salt Lake City and all the discussions we had.

Many thanks go to my present and former colleagues of the AG Bux for the best working atmosphere I have experienced so far. I have always enjoyed being part of this group.

I'm grateful to Benjamin Brück, Jonas Flechsig, Christoph Hilmes, Dawid Kielak, Thomas Lessmann, and Yuri Santos Rego for proofreading parts of this thesis.

Meiner Familie möchte ich für die ständige Unterstützung danken. Dafür, dass sie mir stets den Rücken frei gehalten hat und mir damit die Möglichkeit gab, mich auf die Anfertigung dieser Arbeit zu konzentrieren.

Ein besonderer Dank gilt meiner Freundin Cynthia. Ich bin dafür dankbar, dass sie mit mir die schönsten und dunkelsten Momente geteilt hat. Vor allem möchte ich mich für ihr Verständnis bedanken. Dafür, dass sie es ausgehalten hat, dass ich so häufig nicht da war.

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1 INTRODUCTION

1.1 FINITENESS PROPERTIES

There was a time in the history of group theory where it was not clear whether there are finitely generated groups that are not finitely presented. This changed in 1937 when B. Neumann showed that there are uncountably many finitely generated groups [27]. Since there are only countably many finitely presented groups, he proved the existence of finitely generated groups that are not finitely presented. In the following decades it became clear that such groups do not have to be exotic or pathological. Many natural constructions like the wreath product and the amalgamated product can be easily applied to produce finitely generated groups that are not finitely presented.

In 1965 Wall introduced the finiteness properties F_n which generalize the properties of being finitely generated and finitely presented [32]. A group G is type F_n if it acts freely on a contractible cell complex X such that the quotient complex $G\backslash X$ has compact n-skeleton. By looking at the action of a group G on its Cayley graph, respectively on its Cayley complex, it becomes clear that G is of type F_1, respectively of type F_2, if and only if G is finitely generated, respectively finitely presented.

After the introduction of the finiteness properties F_n the natural question arose whether for every $n \in \mathbb{N}$ there is a group of type F_n but not F_{n+1}. The first example that went beyond finite presentability was found by Stallings in 1963 [30] (even before the properties F_n were defined). He showed that the group

$$G := \langle a, b, c, x, y \colon [x, a], [y, a], [x, b], [y, b], [a^{-1}x, c], [a^{-1}y, c], [b^{-1}a, c] \rangle,$$

which he constructed as an iterated amalgamated product, has an infinitely generated third homology group $H_3(G; \mathbb{Z})$ which implies that it cannot be of type F_3. In 1976 Bieri realized that G is the second member of a series of groups $(G_n)_{n \in \mathbb{N}}$ where G_n is of type F_n but not of type F_{n+1} [7]. He defined these groups as the kernels

$$G_n := \ker \Big(\prod_{i=1}^{n} F(a_i, b_i) \xrightarrow{\phi} \mathbb{Z} \Big),$$

where $F(a_i, b_i)$ is the free group with basis $\{a_i, b_i\}$ and ϕ is the morphism that sends all basis elements a_i, b_i to 1.

A further major breakthrough was achieved in 1997 by Bestvina and Brady in their seminal paper [6] in which they show that the properties F_n do not coincide with their homological counterparts

FP_n, which are defined via projective resolutions. They replace the product $\prod_{i=1}^{n} F(a_i, b_i)$ in Bieris construction by arbitrary right-angled Artin groups A_Γ and show that the finiteness properties of the groups

$$BB_\Gamma := \ker(A_\Gamma \xrightarrow{\phi} \mathbb{Z})$$

can be translated to connectivity properties of the flag complex of Γ.

Similar achievements were obtained by Abels and Brown in 1985 in the theory of solvable groups. They studied the finiteness properties of the subgroups $A_n(p) < \mathrm{GL}_{n+1}(\mathbb{Z}[1/p])$, where p is an arbitrary prime, consisting of upper triangular matrices $(a_{i,j})$ with $a_{1,1} = a_{n+1,n+1} = 1$ and showed the following.

Theorem 1.1. *For every $n \in \mathbb{N}$ and every prime $p \in \mathbb{P}$ the group $A_n(p)$ is of type F_{n-1} but not F_n.*

It is not hard to see that the finiteness properties of $A_n(p)$ are the same of those of the subgroup of $\mathrm{SL}_{n+1}(\mathbb{Z}[1/p])$ consisting of upper triangular matrices with $a_{1,1} = a_{n+1,n+1}$. The groups $A_n(p)$, nowadays called Abels groups, motivated many results of this thesis. To state some of these results let us fix a bit of notation.

Definition 1.2. Let R be a unital ring. The subgroup of upper triangular matrices in $\mathrm{SL}_{n+1}(R)$ will be denoted by $B_n(R)$. The subgroup of unipotent matrices in $B_n(R)$ will be denoted by $U_n(R)$.

Particularly interesting in the context of finiteness properties are rings of S-integers, such as $\mathbb{Z}[1/N]$. It is well-known, and also follows from our results, that $B_n(\mathbb{Z}[1/N])$ is of type F_n for every $n \in \mathbb{N}$. In this case we also say that a group is of type F_∞. On the other hand it is an easy exercise to show that $U_n(\mathbb{Z}[1/N])$ is not even finitely generated if $N \notin \mathbb{Z}^\times$.

One of the main goals of this thesis is to find out what happens between these groups.

Question 1.3. *Given any group $U_n(\mathbb{Z}[1/N]) \leq H \leq B_n(\mathbb{Z}[1/N])$, what are the finiteness properties of H?*

Let us first convince ourselves that there are some interesting phenomena concerning finiteness properties that happen between $U_n(\mathbb{Z}[1/N])$ and $B_n(\mathbb{Z}[1/N])$.

Example 1.4. For every prime $p \in \mathbb{P}$ the group

$$H_1 := \left\{ \begin{pmatrix} p^k & * & * \\ 0 & p^{-2k} & * \\ 0 & 0 & p^k \end{pmatrix} \in \mathrm{SL}_3(\mathbb{Z}[1/p]) \colon k \in \mathbb{Z} \right\}$$

is finitely generated but not finitely presented.

Finite generation can be easily checked by hand. To see that H_1 is not finitely presented one can check that the quotient of H_1 by its center is not finitely presented, which again can be done by hand. Hence the claim follows from the fact that metabelian quotients of finitely presented solvable groups are finitely presented [11].

The group H_1, as well as each group H_i in the following examples, is a special case of Corollary 10.30.

Example 1.5. For every prime $p \in \mathbb{P}$ the group

$$H_2 := \left\{ \begin{pmatrix} p^k & * & * \\ 0 & 1 & * \\ 0 & 0 & p^{-k} \end{pmatrix} \in \mathrm{SL}_3(\mathbb{Z}[1/p]) \colon k \in \mathbb{Z} \right\}$$

is of type F_∞.

Example 1.6. Let $p, q \in \mathbb{P}$ be two different primes. The group

$$H_3 := \left\{ \begin{pmatrix} p^k q^{-k} & * \\ 0 & p^{-k} q^k \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}[1/pq]) \colon k \in \mathbb{Z} \right\}$$

is finitely generated but not finitely presented.

Example 1.7. Let $p, q \in \mathbb{P}$ be two different primes. The group

$$\left\{ \begin{pmatrix} p^{k_1} q^{l_1} & * & * \\ 0 & p^{k_2} q^{l_2} & * \\ 0 & 0 & p^{k_3} q^{l_3} \end{pmatrix} \in \mathrm{SL}_3(\mathbb{Z}[1/pq]) \colon k_1 + l_1 + 2l_2 = 3l_3 + k_3 \right\}$$

is of type F_3 but not F_4. It will be denoted by H_4.

The following theorem was derived by Witzel [34] from the work of Bux and Wortman on the connectivity of horospheres in Euclidean buildings [16].

Theorem 1.8. *Let $p \in \mathbb{P}$ be a prime, let $n \in \mathbb{N}$, and let $d_1 \geq d_2 \geq \ldots \geq d_{n+1}$ be integers, not all equal. We consider the morphism*

$$\phi \colon B_n(\mathbb{Z}[1/p]) \to \mathbb{Z}, (a_{i,j}) \mapsto \sum_{i=1}^{n} d_i v_p(a_{i,i})$$

where v_p denotes the p-adic valuation. The group $\Gamma := \ker(\phi)$ is of type F_{n-1} but not of type F_n.

In order to formulate our main result on Question 1.3 we have to introduce a set of canonical homomorphisms $B_n(\mathbb{Z}[1/N]) \to \mathbb{Z}$.

Definition 1.9. Let $n, N \in \mathbb{N}$ be arbitrary natural numbers. For every $1 \leq k \leq n$ and every prime factor p of N we define the morphism

$$\chi_{k,p} \colon B_n(\mathbb{Z}[1/N]) \to \mathbb{Z}, (a_{i,j}) \mapsto v_p(a_{k+1,k+1}) - v_p(a_{k,k}),$$

where v_p denotes the p-adic valuation. Let $\mathcal{B}_{n,N}$ denote the union of these morphisms.

Example 1.10.

$$\chi_{1,2}\begin{pmatrix} 1/4 & 3 & 5 \\ 0 & 6 & 6 \\ 0 & 0 & 2/3 \end{pmatrix} = v_2(6) - v_2(1/4) = 1 - (-2) = 3.$$

Theorem 1.11. *Let $k, n, N \in \mathbb{N}$ be natural numbers and let*

$$U_n(\mathbb{Z}[1/N]) \leq H \leq B_n(\mathbb{Z}[1/N])$$

be a subgroup. Suppose that every prime factor $p|N$ satisfies $p \geq 2^{n-1}$. Then H is of type F_k if and only if there is no non-trivial homomorphism of the form $\chi = \sum\limits_{i=1}^{k} \lambda_i \chi_i$ with $\lambda_i \geq 0$ and $\chi_i \in \mathcal{B}_{n,N}$, that vanishes on H.

This theorem is a special case of Corollary 10.30 which considers Borel groups in Chevalley groups of classical type. Note that the character ϕ in Theorem 1.8 can be written as $\sum_{i=1}^{n} \lambda_i \chi_{i,p}$ with $\lambda_i \geq 0$, using that $\sum_{i=1}^{n+1} v_p(a_{i,i}) = 0$. Conversely, not every character $\sum_{i=1}^{n} \lambda_i \chi_{i,p}$ with $\lambda_i > 0$ is an admissible ϕ in Theorem 1.8. Thus Theorem 1.11 exhibits subgroups of $B_n(\mathbb{Z}[1/p])$ of type F_{n-1} but not of type F_n that are not covered by Theorem 1.8.

In practice it happens quite often that groups with interesting finiteness properties appear as kernels of some naturally given ambient group. Recall for example that this is the case for the Bestvina-Brady groups BB_Γ and for the groups Γ_n. Furthermore it is an easy observation that the groups H_1, H_2, H_3, and H_4 from the examples above appear as kernels of appropriate homomorphisms $B_n(\mathbb{Z}[1/N]) \to \mathbb{Z}$. Note for example that

$$H_4 = \ker(\chi_{1,p} + \chi_{2,p} + \chi_{1,q} + 3\chi_{2,q}).$$

This suggests that there might be a theory that can describe the finiteness properties of kernels of morphisms to abelian groups. Such a theory is provided by the theory of Σ-invariants.

1.2 Σ-INVARIANTS

As the title suggests, this thesis is about Σ-invariants. These invariants are also known as BNSR-invariants where the letters represent the creators of these invariants, namely Bieri, W. Neumann, Strebel, and Renz.

Originally, Σ-invariants were only defined for metabelian groups [12] in order to answer a question of G. Baumslag. He asked how finitely presented metabelian groups can be distinguished from finitely generated metabelian groups that are not finitely presented [5]. The definition in [12] involves valuations of modules over finitely generated abelian groups that come from the structure of metabelian groups.

After a series of generalizations of this first version of the Σ-invariant (see e.g. [9], [14], and [25]), the version of Σ-invariants which will be studied in this thesis was defined by Bieri and Renz in [10].

Unlike most invariants in group theory, the Σ-invariants of a group are not algebraic structures themselves but rather appear as geometric structures on the character sphere of the group.

Definition 1.12. Let G be a finitely generated group. The space

$$S(G) = (\operatorname{Hom}(G, \mathbb{R}) \setminus \{0\})/\sim$$

is called the *character sphere* of G where $\chi \sim \psi$ if $\exists \lambda > 0$ s.t. $\lambda\chi = \psi$.

The nth Σ-invariant of a group G, denoted by $\Sigma^n(G)$, is a certain subset of $S(G)$ that satisfies the following property which was proven by Renz in his thesis [28].

Theorem 1.13. *Let G be a group of type F_n and let $[G,G] \leq H \leq G$ be a subgroup. Then*

$$H \text{ is of type } F_n \Leftrightarrow \{\chi \in S(G) \colon \chi(H) = 0\} \subset \Sigma^n(G).$$

For a precise definition of Σ-invariants we refer to 2.46. Note that for a given group G of type F_n, Theorem 1.14 implies that the information about the finiteness properties F_k, where $k \leq n$, of all kernels $\ker(G \to \mathbb{Z})$ lies in $\Sigma^k(G)$. In particular we could read off the finiteness properties of the groups BB_Γ and Γ_n from the Σ-invariants of A_Γ and $B_n(\mathbb{Z}[1/p])$ once they are known.

This also indicates that computing the Σ-invariants of a group is very difficult in general. In fact there are not many examples of groups available of which all Σ-invariants are known.

In the case of right-angled Artin groups a full computation of its Σ-invariants was achieved by Meier, Meinert, and VanWyk in 1998 [24] and reproved by Bux and Gonzalez with more geometric methods [15].

The main goal of this thesis was the full computation of the Σ-invariants of the S-arithmetic subgroups of Borelgroups in Chevalley groups. The most prominent such groups are $B_n(\mathbb{Z}[1/N])$. In this special case the main result of this thesis can be stated as follows.

Theorem 1.14. *Let $n, N \in \mathbb{N}$ be natural numbers. Suppose that every prime factor $p|N$ satisfies $p \geq 2^{n-1}$. Then the Σ-invariants of the group $B_n(\mathbb{Z}[1/N])$ are given by*

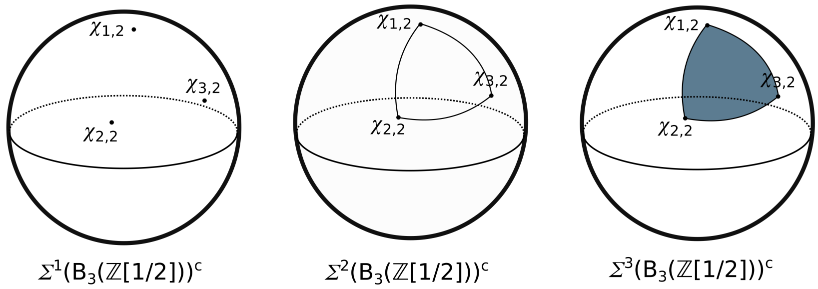
$$\Sigma^k(B_n(\mathbb{Z}[1/N])) = S(B_n(\mathbb{Z}[1/N])) \setminus \Delta^{(k)} \text{ for every } k \in \mathbb{N},$$

where $\Delta^{(k)}$ denotes the k-skeleton of the simplex $\Delta \subset S(B_n(\mathbb{Z}[1/N]))$ that is the convex hull of $\mathcal{B}_{n,N}$.

This partially confirms the following conjecture of my supervisor Stefan Witzel.

Conjecture 1.15. *Theorem 1.15 holds without any restrictions on N.*

Example 1.16. The complements of the Σ-invariants of the group $B_3(\mathbb{Z}[1/2])$, denoted by $\Sigma^n(B_3(\mathbb{Z}[1/2]))^c$, are given as follows:



1.3 EUCLIDEAN BUILDINGS

In order to prove Theorem 1.15 we will consider the action of the group $B_n(\mathbb{Z}[1/N])$ on some appropriate Euclidean building X. In Chapter 10 we will see that this action is cocompact and that the cell stabilizers are of type F_∞. This will allows us to translate the problem of determining the Σ-invariants of $B_n(\mathbb{Z}[1/N])$ to a geometric problem in X.

Let A be an apartment in X, let $\sigma \subset \partial_\infty A$ be a chamber at infinity, and let $\rho = \rho_{\sigma,A} \colon X \to A$ denote the retraction from infinity associated to σ and A. We will consider the space of functions

$$X^*_\sigma = \{\alpha \circ \rho \colon \alpha \in A^*\}$$

where A^* denotes the dual space of A. The subspaces of X we are going to study are of the form $X_{h\geq r} := h^{-1}([r,\infty))$ where $h \in X^*_\sigma$. In order to determine the Σ-invariants of $B_n(\mathbb{Z}[1/N])$ we have to solve the following question.

Question 1.17. *Given any function $h \in X^*_\sigma$, any $k \in \mathbb{N}_0$, and any $r \in \mathbb{R}$. Is there an $s \leq r$ such that the canonical morphisms*

$$\pi_k(\iota) \colon \pi_k(X_{h\geq r}, x) \to \pi_k(X_{h\geq s}, x)$$

are trivial for every $x \in X_{h\geq r}$?

The largest part of this thesis concerns the study of this question. This study essentially splits into three parts. The first part is about proving a positive answer to Question 1.17 under certain circumstances. The idea is to introduce spaces Z that lie between two levels, i.e. that satisfy $Z \subset X_{r\geq h\geq s}$ and glue them on the space $X_{h\geq r}$. Then we will filter the space $Z \cup X_{h\geq r}$ with the help of some combinatorial Morse function. These arguments are given in Chapters 3 and 4. Questions like 1.17 appear naturally in the theory of Σ-invariants and usually a proof that provides a confirmative answer to that question already shows that the spaces $X_{h\geq r}$ themselves are k-connected. It is worth mentioning that the proof of the positive direction as given in this thesis is, as far as I know, the first one in the theory of Σ-invariants that takes advantage of the choice of s.

The second part is about proving a negative answer to Question 1.17 under certain circumstances. In this part we will develop a method that allows us to take cycles from infinity and pull them into the space. This technique will be explained in Chapter 5 and is interesting in its own right.

The third part of the study of the spaces $X_{h\geq r}$ is about a reduction process that allows us to restrict ourselves to the cases in the first two parts.

We will apply a construction for CAT(0)-spaces, which was introduced by Caprace in [17], to Euclidean buildings. More precisely we will take a vertex at infinity $\xi \in \partial_\infty X$ and consider the space of all rays $[x, \xi)$ converging to ξ. By identifying two rays if they share a common point in X, we obtain a quotient space X^ξ that carries a canonical structure of a Euclidean building. This procedure will be described in Chapter 7.

2 BASIC NOTIONS

A large part of this work deals with group actions on spherical and Euclidean buildings. These are certain cell complexes that carry a natural metric and appear in the study of Chevalley groups which will be introduced in Chapter 9. In this section we introduce the necessary background to deal with buildings. Furthermore, we will introduce Σ-invariants of groups whose determination in the case of S-arithmetic subgroups of Borel groups is the main goal of this work.

2.1 METRIC SPACES

In order to introduce $\mathrm{CAT}(0)$-spaces we start by recalling the notions of geodesic segments, rays, and spaces.

Definition 2.1. Let X be a metric space. A *geodesic segment* in X is an isometric embedding $\gamma\colon [0,l] \to X$. A *geodesic ray* in X is an isometric embedding $\gamma\colon [0,\infty) \to X$.

Notation 2.2. For convenience, we will often identify geodesic segments and rays with their images.

Definition 2.3. A metric space X is called *geodesic* if every two points of X can be connected by a geodesic segment.

Given a geodesic metric space X and two points x, y, we will denote by $[x,y]$ a choice of a geodesic segment from x to y. This notation should not lead to ambiguity since in most spaces we are going to consider, geodesic segments will be unique.

One of the most prominent classes of geodesic metric spaces is the class of $\mathrm{CAT}(0)$-spaces. To define $\mathrm{CAT}(0)$-spaces we have to recall the notion of geodesic triangles and their comparison triangles in the standard Euclidean space $\mathbb{E}^2 := (\mathbb{R}^2, \langle\cdot,\cdot\rangle)$.

Definition 2.4. Let (X,d) be a geodesic metric space. A *geodesic triangle* in X consists of three points $x,y,z \in X$ together with a choice of geodesic segments $[x,y]$, $[y,z]$ and $[z,x]$ connecting these points. Such a triangle will be denoted by $\Delta(x,y,z)$. One can show that, up to congruence, there is a unique geodesic triangle $\Delta(\overline{x},\overline{y},\overline{z}) \subset \mathbb{E}^2$ such that

$$\| \overline{u} - \overline{v} \| = d(u,v)$$

for every two points $u,v \in \{x,y,z\}$. Such a triangle is called a *comparison triangle* of $\Delta(x,y,z)$ in $\mathbb{E}^2$. For every two points $u,v \in \{x,y,z\}$ and every point $a \in [u,v]$ we define its *comparison point* in $[\overline{u},\overline{v}]$, denoted by $\overline{a}$, to be the unique point in $[\overline{u},\overline{v}]$ with $d(u,a) = d(\overline{u},\overline{a})$.

One can think of CAT(0)-spaces as those geodesic metric spaces in which geodesic triangles cannot be thicker than their corresponding Euclidean comparison triangles. The following definition gives a precise meaning to this.

Definition 2.5. A geodesic metric space (X, d) is a CAT(0)-*space* if for every two points a, b in every geodesic triangle $\Delta(x, y, z) \subset X$ the corresponding comparison points $\overline{a}, \overline{b}$ in some comparison triangle $\Delta(\overline{x}, \overline{y}, \overline{z}) \subset \mathbb{E}^2$ satisfy the inequality

$$d(a, b) \leq \| \overline{a} - \overline{b} \| .$$

Remark 2.6. If we replace the comparison triangles in $\mathbb{R}^2$ with comparison triangles on the unit sphere S^2, we obtain the notion of a CAT(1)-*space*.

In the following we will work with the boundary at infinity of CAT(0)-spaces. To define the boundary at infinity, we have to introduce a relation on the set of geodesic rays.

Definition 2.7. Let X be a CAT(0)-space. Two geodesic rays α, β in X are equivalent if there is a constant $c \geq 0$ such that $d_X(\alpha(t), \beta(t)) \leq c$ for every $t \in [0, \infty)$. The space of these classes, denoted by $\partial_\infty X$, is called the *boundary at infinity of* X.

In the case of a complete CAT(0)-space X there is a canonical way of identifying $\partial_\infty X$ with the set of geodesic rays emanating from an arbitrary fixed point $x \in X$ (see [13, Proposition II.8.2]).

Proposition 2.8. *Let X be a complete* CAT(0)*-space and let $\gamma\colon [0, \infty) \to X$ be a geodesic ray. For every point $x \in X$ there is a unique geodesic ray $\gamma'\colon [0, \infty) \to X$ issuing from x that is equivalent to γ.*

In the situation of Proposition 2.8 we denote the geodesic ray issuing from a point $x \in X$ and representing a point at infinity $\xi \in \partial_\infty X$ by $[x, \xi)$. For such a point ξ it will be useful to have a sort of height function $X \to \mathbb{R}$ that increases when moving towards ξ. Such a height function is provided by the following definition.

Definition 2.9. Let X be a CAT(0)-space, $\xi \in \partial_\infty X$, and $p \in X$. The *Busemann function* associated to ξ and p is given by

$$\beta_{\xi,p}\colon X \to \mathbb{R}, x \mapsto \lim_{t\to\infty}(t - d(x, [p, \xi)(t))).$$

Remark 2.10. It can be easily seen that the limit $\lim\limits_{t\to\infty}(t - d(x, [p, \xi)(t)))$ exists and that the difference $\beta_{\xi,p} - \beta_{\xi,q}$ is constant for $p, q \in X$. Some authors prefer to define Busemann functions corresponding to ξ and p by $x \mapsto \lim\limits_{t\to\infty}(d(x, [p, \xi)(t)) - t)$. Clearly this only changes the sign of the Busemann function.

2.2 POLYTOPES AND POLYHEDRA

We quickly recall some basic aspects of the theory of polyhedral cell complexes. The details of the constructions can be found in [13, p. I.7.]. For the rest of this section we fix a metric cell complex X whose cells are isometric to *Euclidean polysimplices*, i.e. products of simplices in some Euclidean space. Further, we will assume that, up to isometry, there are only finitely many cells in X. In order to deal with polysimplices it is sometimes useful to note that they are simple polytopes.

Definition 2.11. A d-dimensional convex polytope C is called *simple* if every vertex in C is contained in exactly d *facets*, i.e. faces of codimension 1.

From this characterization of simple polytopes it follows immediately that products of simple polytopes are again simple. In particular we see that polysimplices are simple. In view of this observation the characterization of simple polytopes given in [35, Proposition 2.16] tells us the following.

Lemma 2.12. *Let A be a face of a polysimplex C of codimension k. Then there are precisely k facets of C that contain A. In particular, there is a unique set of facets of C that intersects in A.*

From now on, we will use the following notation.

Notation 2.13. Unless otherwise stated, the term *cell* will always be used to denote the relative interior of its ambient closed polytope. Nevertheless we will say that a cell A is a *face* of a cell B if A is contained in $\overline{B}$.

This has the advantage that for every $x \in X$ there is a unique cell containing x.

Definition 2.14. Let Z be a subcomplex of X and let A be a cell in Z. The *relative star* of A in X with respect to Z, denoted by $\mathrm{st}_Z(A)$, is the union of cells $B \subset Z$ such that A is a face of B. In this case B is said to be a *coface* of A. For an arbitrary point $x \in A$ we further define $\mathrm{st}_Z(x) = \mathrm{st}_Z(A)$. We will omit the subscript in the case where Z coincides with X.

Notation 2.15. Let A be a cell in X. The *boundary* ∂A of A is the complex of proper faces of A. In particular we have $\partial(v) = \emptyset$ for each vertex v of X. In the following we will use the convention $\dim(\emptyset) = -1$.

By definition we can embed every polysimplex isometrically as a convex subspace of some Euclidean vector space. This allows us to speak about angles between geodesic segments that are contained in a common cell and emanate from the same point.

Definition 2.16. Let Z be a subcomplex of X and let $x \in Z$ be a point. The *relative link* of x with respect to Z, denoted by $\mathrm{lk}_Z(x)$, is the set of *directions* at x, i.e. germs of geodesics emanating from x, that point into Z. Let further A be a cell in Z. The *relative link* of A in X with respect to Z, denoted by $\mathrm{lk}_Z(A)$, is the set of directions emanating from the barycenter $\mathring{A}$ of A that are orthogonal to A and point into Z. If Z coincides with X we will just speak about the *link* of x (respectively A) and write $\mathrm{lk}(x)$ (respectively $\mathrm{lk}(A)$).

It will be useful to decompose $\mathrm{lk}(x)$ into subspaces of the form $\mathrm{lk}_{\overline{B}}(x)$ where B is a cell in the star of x. We endow each such space $\mathrm{lk}_{\overline{B}}(x)$ with the metric that is given by the angle between directions. In order to define the distance between two arbitrary points $p, p' \in \mathrm{lk}(x)$ we consider the set $\Gamma(p, p')$ of all maps $\gamma\colon [0, t] \to \mathrm{lk}(x)$ such that there is a subdivision $0 = t_0 \leq \ldots \leq t_n = t$ with the property that each restriction $\gamma_{|[t_i,t_{i+1}]}$ is a geodesic segment in a space of the form $\mathrm{lk}_{\overline{B_i}}(x)$ for some cell $B_i \subset \mathrm{st}(x)$. In this case we define $l(\gamma) = t$. The discussion in [13, p. I.7.38.] tells us that the map

$$(p, p') \mapsto \inf_{\gamma \in \Gamma(p,p')} l(\gamma)$$

defines a metric on $\mathrm{lk}(x)$. Let A be a face of B. The following well known construction will help us to decompose $\mathrm{lk}_{\overline{B}}(\mathring{A})$ into its subspaces $\mathrm{lk}_{\overline{A}}(\mathring{A}) \cong \partial A$ and $\mathrm{lk}_{\overline{B}}(A)$.

Definition 2.17. Let Y and Z be two topological spaces. The *join* of Y and Z, denoted by $Y * Z$, is defined to be the quotient space $(Y \times Z \times [0,1])/\sim$ where $(y, z, 0) \sim (y, z', 0)$ and $(y, z, 1) \sim (y', z, 1)$ for every $y, y' \in Y$ and $z, z' \in Z$.

Let x be a point in X and let A be the cell containing x. By choosing $\delta > 0$ small enough we can ensure that the δ-neighborhood $B(x, \delta)$ of x is contained in the star of A. Let $\varepsilon := \frac{\delta}{\sqrt{2}}$ and let $U := A \cap B(x, \varepsilon)$ be the open ε-neighborhood of x in A. For a given coface B of A let V denote the set of all $y \in \overline{B} \cap B(x, \varepsilon)$ such that $[x, y]$ is orthogonal to A. This gives us $U \times V \subset B(x, \delta) \cap \overline{B}$ and we see that $U \times V$ a neighborhood of x in $\overline{B}$. From the construction we know that U is the open cone of ∂A over x and that V is the open cone of $\mathrm{lk}_{\overline{B}}(A)$ over x. On the other hand, the space $\overline{B} \cap B(x, \varepsilon)$ can be identified with the open cone over $\mathrm{lk}_{\overline{B}}(x)$. In this situation [13, Proposition I.5.15.] tells us that $\mathrm{lk}_{\overline{B}}(x)$ can be decomposed as

$$\mathrm{lk}_{\overline{B}}(x) \cong \partial A * \mathrm{lk}_{\overline{B}}(A).$$

By applying this observation to every cell in $\mathrm{st}_Z(A)$ we obtain the following.

Lemma 2.18. *Let Z be a subcomplex of X. For every cell $A \subset Z$ there is a canonical homeomorphism*

$$\partial(\mathrm{st}_Z(A)) \to \partial A * \mathrm{lk}_Z(A).$$

2.3 TOPOLOGY

The following definition will often be convenient for us.

Definition 2.19. A d-dimensional cell complex X is called *spherical* if it is $(d-1)$-connected.

We will use the following standard gluing lemma.

Lemma 2.20. *Let $n \in \mathbb{N}_0$ and let Z be a cell complex that can be written as a union of subcomplexes $Z = X \cup \bigcup_{i \in I} Y_i$ where I is an index set. Assume that*

1. *each Y_i is contractible,*
2. *$Y_i \cap Y_j \subseteq X$, and that*
3. *$Y_i \cap X$ is $(n-1)$-connected.*

Then the pair (Z, X) is n-connected. The same holds if "n-connected" is replaced by "n-acyclic".

Proof. To prove the first claim we have to show that for each $0 \leq k \leq n$ every map $(B^k, S^{k-1}) \to (Z, X)$ is homotopic relative S^{k-1} to a map whose image lies in X. Thus for $k = 0$ it suffices to check that each point $p \in Z$ can be connected by a path to a point in X. But this is clear since each Y_i is path-connected by (1) and its intersection with X is non-empty by (3). Note that this allows us to restrict ourselves to the case where X and Z are path-connected. For $k = 1$ the homotopy part of the claim follows from the van Kampen theorem. In view of Hurewicz's theorem it remains to show that the relative homology groups $\widetilde{H}_k(Z, X)$ vanish for $1 \leq k \leq n$. Since taking colimits commutes with the homology functor (see e.g. [23, Theorem 14.6.]) it follows from assumption (3) that it is sufficient to consider the case where $I = \{i\}$ is a singleton. We write $Y := Y_i$ and consider the part

$$0 = \widetilde{H}_k(Y) \to \widetilde{H}_k(Y, Y \cap X) \to \widetilde{H}_{k-1}(Y \cap X) \to \widetilde{H}_{k-1}(Y) = 0$$

of the long exact sequence for the pair $(Y, Y \cap X)$. By (2) we see that $\widetilde{H}_k(Y, Y \cap X) \cong \widetilde{H}_{k-1}(Y \cap X) = 0$ for $k \leq n$. Since $Z = X \cup Y$ it remains to observe that excision gives us $\widetilde{H}_k(Z, X) \cong \widetilde{H}_k(Y, Y \cap X)$. □

The following result tells us that the topological properties of cell complexes behave well under taking joins (see e.g. [26, Lemma 2.3.]).

Lemma 2.21. *Let X and Y be two cell complexes. If X is m-connected and Y is n-connected then their join $X * Y$ is $(m+n+2)$-connected.*

In order to show that certain complexes are not contractible we will produce several topdimensional chains that have the same boundary. The following easy observation tells us that this is already sufficient.

Lemma 2.22. *Let X be a contractible cell complex of dimension d. Let $z \in Z_{d-1}(X;R)$ be a cycle of dimension $d-1$. Then there is a unique d-chain $b \in C_d(X;R)$ such that $\partial b = z$.*

Proof. First, observe that $Z_d(X;R) = 0$ since there are no cells of dimension $d+1$ and $\widetilde{H}_d(X;R) = 0$. Suppose that there are d-chains B, B' such that $\partial B = \partial B' = Z$. Then $\partial(B - B') = 0$ and therefore $B - B' \in Z_d(X;R) = 0$. □

The following notation will be useful in order to deal with subspaces of cell complexes that are not necessarily subcomplexes.

Notation 2.23. Let X be a cell complex and let $M \subset X$ be an arbitrary subset. The largest subcomplex of X contained in M will be denoted by $X(M)$. We will say that $X(M)$ is the *subcomplex of X supported by M*.

2.4 COXETER COMPLEXES AND BUILDINGS

The spaces we are going to look at will mainly be subcomplexes of spherical and Euclidean buildings. Most of the time we will think of spherical (resp. Euclidean) buildings as metric spaces, more precisely as CAT(1)-spaces (resp. CAT(0)-spaces). In order to obtain this metric we will think of a spherical Coxeter complex as the standard sphere S^d of the appropriate dimension whose simplicial structure is given by the hyperplane arrangement associated to the Coxeter group. Analogously, we will think of a Euclidean Coxeter complex as the standard Euclidean space which is partitioned into bounded cells by a locally finite arrangement of hyperplanes. It will be convenient to have a decomposition of Coxeter complexes in their irreducible parts (see [29, Proposition 1.15.]).

Lemma 2.24. *Every spherical Coxeter complex Σ decomposes as a join*

$$\Sigma = *_{i=1}^{n} \Sigma_i$$

of irreducible, isometrically embedded, pairwise orthogonal, spherical Coxeter complexes Σ_i. Every Euclidean Coxeter complex Σ decomposes as a product

$$\Sigma = \prod_{i=1}^{n} \Sigma_i$$

of irreducible, pairwise orthogonal, Euclidean Coxeter complexes Σ_i.

Note that Euclidean Coxeter complexes are polysimplicial but not necessarily simplicial.

Definition 2.25. Let Σ be a Euclidean Coxeter complex and let $\mathcal{H}$ be the corresponding set of hyperplanes. A vertex $v \in \Sigma$ is called *special* if for every hyperplane $H \in \mathcal{H}$ there is a parallel hyperplane $H' \in \mathcal{H}$ such that $v \in H'$.

It is not difficult to show that every Euclidean Coxeter complex has a special vertex.

The following definition of a building is a slight variation of Definition [2, p. 4.1]. The difference is, that we do not require apartments to be irreducible.

Definition 2.26. A *building* is a cell complex Δ that can be expressed as the union of subcomplexes Σ (called *apartments*) satisfying the following axioms:

(B0) Each apartment Σ is a Coxeter complex.

(B1) For every two simplices $A, B \subset \Delta$, there is an apartment Σ containing both of them.

(B2) If Σ_1 and Σ_2 are two apartments containing two cells A and B, then there is an isomorphism $\Sigma_1 \to \Sigma_2$ fixing A and B pointwise.

The building Δ is called *spherical* (respectively *Euclidean*) if its apartments are spherical (respectively Euclidean) Coxeter complexes.

Definition 2.27. Let Δ be a building. A cell $A \subset \Delta$ of maximal dimension is called a *chamber*. A cell $A \subset \Delta$ of codimension 1 is called a *panel*.

One of the main features of Euclidean buildings is that they posses a natural $\mathrm{CAT}(0)$-metric (see [2, Theorem 11.16.]).

Theorem 2.28. *Let X be a Euclidean building and let $d\colon X \times X \to \mathbb{R}$ be the function given by $(x, y) \mapsto d_\Sigma(x, y)$ where Σ is any apartment containing x and y and d_Σ is the Euclidean metric on Σ. Then the function d is a well-defined* $\mathrm{CAT}(0)$*-metric on X.*

The same procedure allows us to view a spherical building as a metric space. In this case the building becomes a $\mathrm{CAT}(1)$-space (see [13, II.10A.4]) but we shall not use this fact. In particular we can speak about geodesics in spherical and Euclidean buildings so that the following definition makes sense.

Definition 2.29. Let X be a Euclidean building and let $A, B \subset X$ be two cells in X. Let further $a \in A$, $b \in B$ be any two points. The *projection* of A to B, denoted by $\mathrm{pr}_B(A)$, is the unique cell that contains an initial part of the open geodesic (a, b). Projections are also defined in spherical buildings. In this case one has to impose the condition that there is no apartment Σ that contains A and B as antipodal faces.

Definition 2.30. Let Δ be an arbitrary building. A finite sequence of chambers $E_1, \ldots, E_n$ in Δ is called a *gallery* if every two consecutive chambers E_i, E_{i+1} share a common panel. In this case we will also write $E_1 | \ldots | E_n$.

In the following we will mainly be interested in thick buildings. These are defined as follows.

Definition 2.31. Let Δ be a building. The *thickness* of Δ, denoted by $\mathrm{th}(\Delta)$, is the largest number such that each panel $P \subset \Delta$ contains at least $\mathrm{th}(\Delta)$ chambers in its star. If there is no such number we will write $\mathrm{th}(\Delta) = \infty$. If $\mathrm{th}(\Delta) \geq 3$ we will just say that Δ is a *thick* building.

An important feature of spherical buildings is that there is a notion of opposition. We say that two points x, y in a spherical building are *opposite* to each other if there is an apartment Σ containing these points such that the antipodal map defined on Σ maps x to y and vice versa. Analogously, we say that two cells are opposite to each other if they are antipodal in some apartment.

Lemma 2.32. *Let Δ be a spherical building and let Σ be an apartment of Δ. For each simplex $A \subset \Sigma$ and every opposite simplex B of A there is an apartment Σ' containing B and the star $\mathrm{st}_\Sigma(A)$ of A.*

Proof. Let A and B be a pair of opposite simplices in Δ. Let $C \subset \mathrm{st}_\Sigma(A)$ be a chamber and let $D \subset \mathrm{st}_\Sigma(A)$ be the opposite chamber of C in $\mathrm{st}_\Sigma(A)$. Let Σ' be an (in fact the unique) apartment containing C and the projection chamber $\mathrm{pr}_B(D)$. From Proposition [2, p. 4.69] it follows that $\mathrm{pr}_A(\mathrm{pr}_B(D)) = D$. Since apartments are closed under taking projections we get $D \subset \Sigma'$. Now the claim follows since the convex hull $\mathrm{conv}(C, D)$ coincides with $\mathrm{st}_\Sigma(A)$. $\square$

2.5 THE SPHERICAL BUILDING AT INFINITY

In this section we will recall the construction of the spherical building at infinity of a Euclidean building.

Definition 2.33. Let X be a Euclidean building and let Σ be an apartment in X. For every special vertex $v \in \Sigma$ and every proper coface $A \subset \mathrm{st}_\Sigma(v)$ let $K_v^\Sigma(A)$ denote the union of all open rays $(v, \xi) := [v, \xi) \setminus \{v\}$ that have an initial segment in A. We say that two such subsets $K_1 := K_{v_1}^{\Sigma_1}(A_1)$ and $K_2 := K_{v_2}^{\Sigma_2}(A_2)$ are equivalent, denoted by $K_1 \sim K_2$, if their boundaries at infinity $\partial_\infty K_1$ and $\partial_\infty K_2$ coincide.

Definition 2.34. Let X be a Euclidean building. The set of subsets $\partial_\infty K_v^\Sigma(A) \subset \partial_\infty X$, where $\Sigma \subset X$ is an apartment, $v \in \Sigma$ is a special vertex, and $A \subset \Sigma$ is a proper coface of v will be denoted by $S_\infty(X)$.

The following result tells us that the boundary $\partial_\infty X$ of a Euclidean building X can be endowed with the structure of a spherical building (see [2, Theorem 11.79.]). This will be especially useful in Chapter 5.

Theorem 2.35. *Let X be a Euclidean building. The space $\partial_\infty X$ can be endowed with the structure of a spherical building where the set of closed* *cells is given by $S_\infty(X)$ and the full system of apartments consists of the boundaries at infinity of the apartments of X.*

For our purposes it will be helpful to have a more flexible version of the subsets $K_v^\Sigma(A)$, in the sense that it allows v to be an arbitrary point rather than a special vertex.

Definition 2.36. Let X be a Euclidean building and let $\sigma \subset \partial_\infty X$ be a simplex at infinity. For each point $p \in X$ let $K_p(\sigma)$ denote the subset of X given by

$$K_p(\sigma) = \bigcup_{\xi \in \sigma} (p, \xi).$$

If σ is a chamber, we say that $K_p(\sigma)$ is a *sector*.

If p is a special vertex of X, then for every sector $K_p(\sigma)$ there is a unique chamber $E \subset \mathrm{st}(p)$ such that $K_p(\sigma) = K_p^\Sigma(E)$ for some appropriate apartment Σ. On the other hand, for every chamber E in $\mathrm{st}(p)$ there is a chamber $\sigma \subset \partial_\infty X$ such that $K_p^\Sigma(E) = K_p(\sigma)$. Note that, in the case of sectors, the equivalence relation given in Definition 2.33 can be written as follows.

Remark 2.37. Two sectors K, K' are equivalent if and only if their intersection $K \cap K'$ contains a sector.

2.6 THE OPPOSITION COMPLEX

An important subcomplex of a spherical building is the complex of chambers which are opposite to a given chamber.

Definition 2.38. Let Δ be a spherical building and let C be a chamber in Δ. The subcomplex of Δ that consists of all cells A that are opposite to some face of C will be denoted by $\mathrm{Opp}_\Delta(C)$.

For our purposes it will be crucial to understand the topological properties of $\mathrm{Opp}_\Delta(C)$. If the spherical building Δ is non-exceptional and thick enough, it is a result of Abramenko (see [1, Theorem B]) that $\mathrm{Opp}_\Delta(C)$ is highly connected.

Theorem 2.39. *Let Δ be an arbitrary building of type A_{n+1}, C_{n+1} or D_{n+1} but not an exceptional C_3 building. Assume that $\mathrm{th}(\Delta) \geq 2^n + 1$ in the A_{n+1} case, respectively $\mathrm{th}(\Delta) \geq 2^{2n+1} + 1$ in the other two cases. Then $\mathrm{Opp}(C)$ is spherical but not contractible for every chamber $C \subset \Delta$.*

Remark 2.40. Recall that the Coxeter groups that appear as the Weyl groups of the root systems of type B_n and C_n coincide. In the theory of buildings it is a common convention to speak of buildings of type C_n in this case.

2.7 Σ-INVARIANTS

Unlike most invariants in group theory, the Σ-invariants of a group are not algebraic structures themselves. Instead they live on the so-called character sphere of a group which consists of equivalence classes of non-trivial characters of the group.

Definition 2.41. Let G be a finitely generated group. A group homomorphism $\chi\colon G \to \mathbb{R}$ is called a *character* of G. Two characters χ and ψ of G are equivalent, denoted by $\chi \sim \psi$, if there is a real number $r > 0$ such that $\chi = r\psi$. The *character sphere* of G, denoted by $S(G)$, is given by the quotient space $(\mathrm{Hom}(G, \mathbb{R}) \backslash \{0\})/\sim$ of classes of non-trivial characters of G.

Note that $\mathrm{Hom}(G, \mathbb{R})$ is a finite dimensional real vector space and thus $S(G)$ carries a natural topology which turns $S(G)$ into a sphere of dimension $\dim(\mathrm{Hom}(G, \mathbb{R})) - 1$.

More generally if V is a finite dimensional real vector space we denote by $S(V)$ the space of positive homothety classes of non-trivial elements of V.

Definition 2.42. Let $(X_\lambda)_{\lambda\in\Lambda}$ be a directed system of cell complexes where $(\Lambda, \leq)$ is a directed poset and let $X_\alpha \xrightarrow{f_{\alpha,\beta}} X_\beta$ be continuous maps for $\alpha \leq \beta$. The system $(X_\lambda)_{\lambda\in\Lambda}$ is *essentially n-connected* for some $n \in \mathbb{N}_0$ if for every index $\alpha \in \Lambda$ there is an index $\beta \in \Lambda$ with $\alpha \leq \beta$ such that the induced maps

$$\pi_k(f_{\alpha,\beta})\colon \pi_k(X_\alpha, x) \to \pi_k(X_\beta, x)$$

are trivial for every $x \in X_\alpha$ and every $0 \leq k \leq n$. Analogously, we say that the system $(X_\lambda)_{\lambda\in\Lambda}$ is *essentially n-acyclic* for some $n \in \mathbb{N}_0$ if for every $\alpha \in \Lambda$ there is a $\beta \geq \alpha$ such that $\widetilde{H}_k(f_{\alpha,\beta}) = 0$ for every $0 \leq k \leq n$.

The notion of essential connectivity appears naturally if one has to deal with group actions that are not cocompact.

Definition 2.43. A group G is said to be of *type* F_n if it acts freely on a contractible cell complex X such that the quotient of the n-skeleton of X by the group action is compact.

In the following, we will often suppress the class of a character by just writing $\chi \in S(G)$. This will not cause any problems since all properties of characters we are going to look at are invariant under scaling with positive real numbers. In order to define the Σ-invariants of a group G we have to extend its characters equivariantly to an appropriate cell complex on which G acts.

Definition 2.44. Let G be a group acting on a topological space X. Let further χ be a character of G. A continuous function $h\colon X \to \mathbb{R}$ is called a *height function associated to* χ if it is equivariant with respect to the action of G on $\mathbb{R}$ via χ, i.e.

$$h(g(x)) = \chi(g) + h(x) \text{ for every } g \in G, x \in X.$$

In the following, we will consider *superlevelsets* in X with respect to h. These are subsets of the form $X_{h\geq r} := h^{-1}([r, \infty))$ for $r \in \mathbb{R}$. Analogously we define $X_{h\leq r}, X_{h=r}$ etc.

If the action is free, then it is always possible to find height functions for characters (see [28, Konstruktion II.2.2.]).

Proposition 2.45. *Let G be a group. Suppose that G acts freely on a contractible cell complex X such that $G\backslash X$ has finite n-skeleton. For every character $\chi\colon G \to \mathbb{R}$ there is a height function $h\colon X \to \mathbb{R}$ associated to χ.*

We are now ready to define what Σ-invariants are.

Definition 2.46. Let G be a group that acts freely on a contractible cell complex X such that the quotient of the n-skeleton of X by the group action is compact. For every character χ of G let h_χ be a height function associated to χ. The nth *Σ-invariant* of G, denoted by $\Sigma^n(G)$, is defined to be the subset of the character sphere that consists of characters χ such that the system $(X_{h_\chi\geq r})_{r\in\mathbb{R}}$ is essentially $(n-1)$-connected.

Note that the invariant $\Sigma^n(G)$ is only defined for groups of type F_n. It can be shown (see [28, Bemerkungen 3.5]) that the definition of Σ-invariants does not depend on the choices made in it. The next theorem, which is a special case of [8, Theorem 12.1], tells us that the assumption of the freeness of the action can be considerably weakened.

Theorem 2.47. *Let G be a group that acts on a contractible cell complex X such that the quotient of the n-skeleton of X by the group action is compact. Suppose that the stabilizer of each p-cell is of type F_{n-p} for $p \leq n-1$. Let χ be a non-trivial character of G. Suppose further that there is a height function $h\colon X \to \mathbb{R}$ associated to χ. Then χ lies in $\Sigma^n(G)$ if and only if the system $(X_{h\geq r})_{r\in\mathbb{R}}$ is essentially $(n-1)$-connected.*

The following result of Bieri and Renz reduces the problem of determining finiteness properties of groups H that sit between some ambient group G and its commutator subgroup $[G, G]$, to the problem of determining the Σ-invariants of G (see [28, Satz C]).

Theorem 2.48. *Let G be a group of type F_n and let $[G, G] \leq H \leq G$ be a subgroup. Then*

$$H \text{ is of type } F_n \Leftrightarrow \{\chi \in S(G) \colon \chi(H) = 0\} \subset \Sigma^n(G).$$

The following notation is often useful to describe the geometry of the subsets $\Sigma^n(G) \subset S(G)$.

Definition 2.49. Let V be a finite dimensional real vector space and let $M \subset S(V)$ be an arbitrary subset. For every $n \in \mathbb{N}$ we define $\operatorname{conv}_n(M) \subset S(V)$ to be the set of elements that are represented by non-trivial vectors of the form $\sum_{i=1}^{n} \lambda_i v_i$ with $\lambda_i \geq 0$ and $[v_i] \in M$. The union of these sets will be denoted by $\operatorname{conv}(M) = \bigcup_{n\in\mathbb{N}} \operatorname{conv}_n(M)$.

3 DECONSTRUCTING SUBCOMPLEXES OF COXETER COMPLEXES

Throughout this section we fix a Euclidean Coxeter complex Σ of dimension d. Let $\Sigma = \prod_{i=1}^{s} \Sigma_i$ be the decomposition of Σ into its irreducible factors. Further, we fix a special vertex $v \in \Sigma$. This allows us to view Σ as a vector space with origin v. Let $\sigma \subset \partial_\infty \Sigma$ be a chamber at infinity and let $E \subset \mathrm{st}_\Sigma(v)$ be the unique chamber that lies in $K_v(\sigma)$. Let further $\{P_1, \ldots, P_d\}$ be the set of panels of E that contain v. For each panel P_i let α_i be a linear form on Σ such that $\alpha_i(P_i) = 0$ and $\alpha_i(E) > 0$. For convenience, we choose α_i so that the set of walls in Σ that are parallel to $\alpha_i^{-1}(0)$ are given by $W_{i,k} := \alpha_i^{-1}(k)$ where $k \in \mathbb{Z}$. Note that for each i there is a unique vertex ξ_i of σ such that the ray $[v, \xi)$ does not lie in $W_{i,0}$. The Busemann function associated to ξ_i and v will be denoted by β_i. It is an easy exercise to see that $\beta_i \colon \Sigma \to \mathbb{R}$ is the linear form characterized by $\beta_i([v, \xi_i)(t)) = t$ and $\beta_i(\xi_i(1)^\perp) = 0$ (see [13, II.8.24.(1)]). Our goal in the next sections will be to study height functions on Euclidean buildings that are given by precomposing linear forms on a fixed apartment with a retraction from infinity onto that apartment. In this section we will study combinatorial properties of the superlevelsets in Σ that come from certain linear forms on Σ. We fix a non-trivial linear form $h \colon \Sigma \to \mathbb{R}$ such that the composition $h \circ [v, \xi) \colon [0, \infty) \to \mathbb{R}$ is strictly decreasing for each $\xi \in \overline{\sigma}$.

Remark 3.1. Let $\eta = \partial_\infty(h^{-1}((-\infty, r]))$ which does not depend on $r \in \mathbb{R}$. The condition that $h \circ \xi$ is strictly decreasing for each $\xi \in \overline{\sigma}$ can also be expressed by saying that $\overline{\sigma} \subseteq \eta^\circ$ where η° denotes the interior of η or equivalently that $\eta \cap \overline{\sigma^{\mathrm{op}}} = \emptyset$.

We will denote by $\sigma = *_{i=1}^{s} \sigma_i$ the join decomposition of σ into its irreducible join factors. Recall that there is a way of projecting simplices at infinity to cells in Σ.

Definition 3.2. Let A be a cell of Σ and let τ be a simplex in $\partial_\infty \Sigma$. The *projection of τ to A*, denoted by $\mathrm{pr}_A(\tau)$, is the unique cell in $\mathrm{st}(A)$ such that for some (equivalently for every) point $\xi \in \tau$ and some (equivalently for every) point $x \in A$, there is an initial segment of (x, ξ) lying in $\mathrm{pr}_A(\tau)$.

The following definition specifies the idea of moving towards a chamber at infinity.

Definition 3.3. Let $\tau \subset \partial_\infty \Sigma$ be a chamber and let $\Gamma = C_1 | \ldots | C_n$ be a gallery in Σ with $C_i \neq C_{i+1}$ for every $1 \leq i < n$. We say that

Γ is τ-*minimal* if for every two consecutive chambers C_i, C_{i+1} of Γ, separated by a panel P, the condition $\mathrm{pr}_P(\tau) = C_{i+1}$ is satisfied. In this case we also say that Γ is *moving towards* τ.

Lemma 3.4. *Let A be a cell in Σ and let τ be a chamber in $\partial_\infty\Sigma$. If $\Gamma = C_1|\ldots|C_n$ is a minimal gallery in $\mathrm{st}(A)$ terminating in $\mathrm{pr}_A(\tau)$ then Γ is τ-minimal.*

Proof. Let $1 \le i \le n-1$ and let P be the panel separating C_i and C_{i+1}. Let W be the wall spanned by P and let R be the half space bounded by W that contains C_{i+1}. Since Γ is minimal and terminating in $\mathrm{pr}_A(\tau)$ it follows that $\mathrm{pr}_A(\tau)$ lies in R. Let $\xi \in \tau$ and $a \in A$ be arbitrary points. By definition of the projection we have $[a,\xi)((0,\varepsilon)) \subset \mathrm{pr}_A(\tau)$ for some $\varepsilon > 0$. In particular we see that the open segment $[a,\xi)((0,\varepsilon))$ is contained in the halfspace R and hence for every point $x \in P$ the translate $[x,\xi)((0,\varepsilon))$ of $[a,\xi)((0,\varepsilon))$ is also contained in R. Thus it follows that $\mathrm{pr}_P(\tau) = C_{i+1}$. $\square$

Lemma 3.4 gives us the following characterization of $\mathrm{pr}_A(\tau)$.

Corollary 3.5. *Let A be a cell in Σ and let τ be a chamber in $\partial_\infty\Sigma$. Let C be a chamber in the star of A. We have $C = \mathrm{pr}_A(\tau)$ if and only if $\mathrm{pr}_P(\tau) = C$ for every panel $A \le P < C$.*

Lemma 3.6. *Let $C \subset \Sigma$ be a chamber and let $I = \{F \le C\colon\ \mathrm{pr}_F\,\sigma = C\}$. There is a unique minimal proper non-empty face in I. In other words there is a face $\emptyset \ne U < C$ such that $A \in I$ if and only if $U \le A \le C$.*

Proof. Let $A, B \in I$ be two cells and let $\mathcal{P}_A$ and $\mathcal{P}_B$ be the sets of panels of C that are cofaces of A respectively B. Corollary 3.5 tells us that $\mathrm{pr}_P(\sigma) = C$ for every $P \in \mathcal{P}_A \cup \mathcal{P}_B$. On the other hand, we know from Lemma 2.12 that $A = \bigcap\limits_{P\in\mathcal{P}_A} P$ and $B = \bigcap\limits_{P\in\mathcal{P}_B} P$ and therefore $A \cap B = \bigcap\limits_{P\in\mathcal{P}_A\cup\mathcal{P}_B} P$. Thus the uniqueness statement in Lemma 2.12 implies that every panel $A\cap B \le P < C$ is contained in $\mathcal{P}_A \cap \mathcal{P}_B$ and therefore satisfies $\mathrm{pr}_P(\sigma) = C$. In view of Corollary 3.5 it remains to show that $A \cap B$ is not empty. To see this let $C = \prod\limits_{i=1}^{s} C_i$ be the decomposition of C into simplices $C_i \subset \Sigma_i$ and let $1 \le j \le s$ be a fixed coordinate. We claim that there is a panel P of C_j such that the corresponding panel

$$C_1 \times \ldots C_{j-1} \times P \times C_{j+1} \ldots \times C_s$$

of C is not contained in $\mathcal{P}_A \cup \mathcal{P}_B$. Indeed, otherwise the ray $[x,\xi)$ would stay in

$$\Sigma_1 \times \ldots \Sigma_{j-1} \times C_j \times \Sigma_{j+1} \ldots \times \Sigma_s$$

for every $x \in C$ and $\xi \in \sigma$. In this case $[x,\xi)$ is constant in the coordinate j which contradicts our assumption that ξ lies in the open chamber σ. Note further that the intersection over a set of panels of a simplex is empty if and only if the set consists of all panels of the simplex. Together with the observation above this shows that $A \cap B = \bigcap_{P \in \mathcal{P}_A \cap \mathcal{P}_B} P$ is not empty. □

Definition 3.7. For each chamber C in Σ we define the *upper face* $C^\uparrow$ of C to be the intersection of all panels $P < C$ such that $\mathrm{pr}_P(\sigma) = C$ (the face U in Lemma 3.6). Analogously, the *lower face* $C^\downarrow$ of C is defined to be the intersection of all panels $P < C$ such that $\mathrm{pr}_P(\sigma) \neq C$ or, which is equivalent, $\mathrm{pr}_P(\sigma^{op}) = C$ where $\sigma^{\mathrm{op}} \subset \partial_\infty\Sigma$ denotes the chamber opposite to σ.

We recall the so-called gate property for Coxeter complexes. See for example [2, Proposition 3.105].

Proposition 3.8. *Let A be a cell of Σ and let C be a chamber of Σ. Then the projection chamber $\mathrm{pr}_A(C)$ has the following property. For every chamber $D \subseteq \mathrm{st}(A)$ the equality*

$$d(D,C) = d(D, \mathrm{pr}_A(C)) + d(\mathrm{pr}_A(C), C)$$

is satisfied.

In particular, there is a minimal gallery from D to C passing through $\mathrm{pr}_A(C)$. Recall from section 2.5 that for each point $x \in \Sigma$ and each simplex $\tau \subset \partial_\infty\Sigma$ we denote by

$$K_x(\tau) = \bigcup_{\xi \in \tau} (x, \xi)$$

the (open) cone corresponding to τ with tip in x.

Remark 3.9. By our choice of $\alpha_1, \dots, \alpha_d$, every sector $K_x(\sigma)$ can be described as the set of points $y \in \Sigma$ such that $\alpha_i(y) > \alpha_i(x)$ for every $1 \leq i \leq d$. Analogously, $K_x(\sigma^{op})$ can be described as the set of points $y \in \Sigma$ such that $\alpha_i(y) < \alpha_i(x)$ for every $1 \leq i \leq d$.

Lemma 3.10. *For every point $x \in \Sigma$ and every $r \in \mathbb{R}$ the intersection*

$$\overline{K_x(\sigma^{op})} \cap h^{-1}((-\infty, r])$$

is compact.

Proof. The polyhedron $\overline{K_x(\sigma^{\mathrm{op}})} \cap h^{-1}((-\infty, r])$ has boundary

$$\begin{aligned}\partial_\infty(\overline{K_x(\sigma^{\mathrm{op}})} \cap h^{-1}((-\infty, r])) &= \partial_\infty(\overline{K_x(\sigma^{\mathrm{op}})}) \cap \partial_\infty(h^{-1}((-\infty, r])) \\ &= \overline{\sigma^{\mathrm{op}}} \cap \eta = \emptyset,\end{aligned}$$

see Remark 3.1. It is therefore compact. □

Note that Lemma 3.10 implies, in particular, that the supported complex

$$\Sigma(\overline{K_x(\sigma^{op})} \cap h^{-1}((-\infty, r]))$$

is also compact.

Lemma 3.11. *Let Z be a bounded subset of Σ. There is a special vertex $w \in \Sigma$ such that Z is contained in the sector $K_w(\sigma^{op})$.*

Proof. Since Z is bounded there is an integer n such that $\alpha_i(z) < n$ for every point $z \in Z$ and every index $1 \leq i \leq d$. In view of Remark 3.9 it suffices to define the vertex $w \in \Sigma$ by $\alpha_i(w) = n$ for all $1 \leq i \leq d$. □

The following lemma provides us with a lower bound for the special vertex in Lemma 3.11 in the case where Z consists of a single point.

Lemma 3.12. *There is a constant $\varepsilon > 0$ such that for every point $x \in \Sigma$ there is a special vertex $w \in \Sigma$ of height $h(w) > h(x) - \varepsilon$ such that the sector $K_w(\sigma^{op})$ contains x.*

Proof. Let $C \subset \Sigma$ be a chamber with $x \in \overline{C}$ and let u_1 be a special vertex of C. We consider the points $z_i = \alpha_i(u_1)$ for every $1 \leq i \leq d$. Let u_2 be the special vertex characterized by $\alpha_i(u_2) = z_i + 1$ for every $1 \leq i \leq d$. Then the subcomplex $\overline{K_{u_1}(\sigma^{op})} \leq \Sigma$ lies in the (open) sector $K_{u_2}(\sigma^{op})$. It follows that the star $\mathrm{st}(u_1)$ is contained in $K_{u_2}(\sigma^{op})$. In particular we see that $x \in \overline{\mathrm{st}(u_1)} \subset \overline{K_{u_2}(\sigma^{op})}$. If we apply the construction a second time we see that the vertex $w \in \Sigma$, characterized by $\alpha_i(w) = z_i + 2$ for every $1 \leq i \leq d$, satisfies the second claim. Let δ_1 be the h-distance between u_1 and u_2 and let δ_2 be the h-diameter of the star of a special vertex. Then by the above construction there is a special vertex $w \in \Sigma$ such that $h(w) \geq h(x) - \varepsilon$ for $\varepsilon = 2\delta_1 + 2\delta_2$ and such that x is contained in $K_w(\sigma^{op})$. □

Definition 3.13. A subcomplex $Z \leq \Sigma$ is called *σ-convex* if for every two cells $A, B \subset Z$ the following is satisfied. Every σ-minimal gallery Γ from $\mathrm{pr}_A(\sigma)$ to $\mathrm{pr}_B(\sigma^{op})$ is contained in Z.

We emphasize that Definition 3.13 does not require the existence of a σ-minimal gallery in Z.

Remark 3.14. Note that we could replace σ by σ^{op} in the definition of σ-convexity.

Definition 3.15. Let Z be a subcomplex of Σ. The *non-separating boundary* of Z, denoted by $R(Z)$, is the union of cells $A \subset Z$ such that $\mathrm{pr}_A(\sigma^{op}) \nsubseteq Z$.

Lemma 3.16. *Let $Z \leq \Sigma$ be a σ-convex subcomplex. The non-separating boundary $R(Z)$ is a subcomplex of Z.*

Proof. Let B be a cell in $R(Z)$ and let A be a face of B. Let Γ be a minimal gallery from $\mathrm{pr}_A(\sigma^{op}) =: C$ to $\mathrm{pr}_B(\sigma^{op}) =: D$. Note that Γ is contained in $\mathrm{st}(A)$ and can be extended to a minimal gallery Γ' from C to $\mathrm{pr}_A(\sigma)$. Indeed, since the chambers $\mathrm{pr}_A(\sigma)$ and $\mathrm{pr}_A(\sigma^{op})$ are opposite in $\mathrm{st}(A)$ this follows from the well-known fact that every chamber in a spherical Coxeter complex is contained in a minimal gallery connecting two given opposite chambers. In particular this implies that Γ is σ-minimal by Lemma 3.4. In order to apply the σ-convexity of Z we note that $C = \mathrm{pr}_{C^\uparrow}(\sigma)$ and $D = \mathrm{pr}_{D^\downarrow}(\sigma^{op})$. Further Lemma 3.6 tells us that B is a coface of $D^\downarrow$. In particular we see that $D^\downarrow$ lies in Z. Suppose that A is not a cell of $R(Z)$. Then by definition we have $C \subseteq Z$ and thus $C^\uparrow \subseteq Z$. Now the σ-convexity of Z implies that the entire gallery Γ is contained in Z. In particular D is a chamber in Z, which is a contradiction to $B \in R(Z)$. Thus we see that A is a cell of $R(Z)$. □

Definition 3.17. Let Z be a subcomplex of Σ. For each chamber C of Z its *σ-length* in Z, denoted by $\ell_Z(C)$, is the length of the longest σ-minimal gallery in Z starting in C. If there are arbitrarily long σ-minimal galleries in Z starting in C, we define $\ell_Z(C) = \infty$.

Lemma 3.18. *Let Z be a σ-convex subcomplex of Σ and let $C \subset Z$ be a chamber with $\ell_Z(C) = 0$. Then the following are satisfied.*

1. $\mathrm{st}(C^\downarrow) \cap Z \subset \overline{C}$.
2. $Z \setminus \mathrm{st}(C^\downarrow)$ *is σ-convex.*
3. $R(Z \setminus \mathrm{st}(C^\downarrow)) = R(Z)$.

Proof. To prove the first claim let A be a cell in $\mathrm{st}(C^\downarrow) \cap Z$ and let $D := \mathrm{pr}_A(\sigma^{op})$. Let further Γ be a minimal gallery from D to C. Note that D is contained in $\mathrm{st}(C^\downarrow)$ and thus that Γ is contained in $\mathrm{st}(C^\downarrow)$. Since $C = \mathrm{pr}_{C^\downarrow}(\sigma^{op})$ it follows form Lemma 3.4 that Γ is σ^{op}-minimal. On the other hand, we have $C = \mathrm{pr}_{C^\uparrow}(\sigma)$ and thus the σ-convexity of Z implies that Γ is contained in Z. Now the condition $\ell_Z(C) = 0$ implies that $C = D$ and thus A is contained in $\overline{C}$.

For the second claim let A and B be two cells in $Z \setminus \mathrm{st}(C^\downarrow)$ such that there is a σ-minimal gallery Γ from $\mathrm{pr}_A(\sigma)$ to $\mathrm{pr}_B(\sigma^{op})$. We have to show that Γ lies in $Z \setminus \mathrm{st}(C^\downarrow)$. By the first claim it thus suffices to show that Γ does not contain C. Suppose that Γ contains C and let Γ' be the subgallery of Γ starting at C. The σ-convexity of Z implies that Γ' is contained in Z and therefore $\ell(\Gamma') = \ell_Z(C) = 0$, i.e. $C = \mathrm{pr}_B(\sigma^{op})$. On the other hand if $C = \mathrm{pr}_B(\sigma^{op})$, then B is a coface of $C^\downarrow$ by Lemma 3.6. But this is a contradiction since there are no cofaces of $C^\downarrow$ lying in $Z \setminus \mathrm{st}(C^\downarrow)$.

To prove the third claim let A be a cell in $R(Z)$. By definition $\mathrm{pr}_A(\sigma^{op}) \not\subseteq Z$ and hence in particular $\mathrm{pr}_A(\sigma^{op}) \not\subseteq Z \setminus \mathrm{st}(C^\downarrow)$. To prove that A is contained in $R(Z \setminus \mathrm{st}(C^\downarrow))$ it suffices to show that $A \not\subseteq \mathrm{st}(C^\downarrow)$.

Otherwise the first claim tells us that A is a coface of $C^{\downarrow}$ lying in $\overline{C}$ and hence by applying Lemma 3.6 we see that

$$\mathrm{pr}_A(\sigma^{op}) = \mathrm{pr}_{C^{\downarrow}}(\sigma^{op}) = C \subset Z.$$

This is a contradiction since by our assumption we have $\mathrm{pr}_A(\sigma^{op}) \nsubseteq Z$. Suppose now that A is a cell in $R(Z \setminus \mathrm{st}(C^{\downarrow}))$. Thus by definition $\mathrm{pr}_A(\sigma^{op}) \nsubseteq R(Z \setminus \mathrm{st}(C^{\downarrow}))$. Then by the first claim either $\mathrm{pr}_A(\sigma^{op}) = C$ or $\mathrm{pr}_A(\sigma^{op}) \nsubseteq Z$. We only have to consider the first case. But in this case Lemma 3.6 tells us again that A is a coface of $C^{\downarrow}$ and hence does not lie in $Z \setminus \mathrm{st}(C^{\downarrow})$. □

An inductive application of Lemma 3.18 provides us with a filtration of compact σ-convex complexes.

Corollary 3.19. *Let Z be a compact σ-convex subcomplex of Σ and let n be the number of chambers in Z. There is a filtration*

$$Z_0 \subsetneq Z_1 \subsetneq \ldots \subsetneq Z_n = Z$$

of Z by subcomplexes Z_i such that

1. *$Z_0 = R(Z)$,*
2. *$Z_{m+1} = Z_m \cup \overline{C_{m+1}}$ for some chamber $C_{m+1} \subseteq Z$ with $\ell_{Z_{m+1}}(C_{m+1}) = 0$, and*
3. *$\mathrm{st}(C^{\downarrow}_{m+1}) \cap Z_{m+1} \subset \overline{C_{m+1}}$.*

Proof. Without loss of generality we may assume that $n > 0$. Indeed, otherwise it follows directly from the definition of the non-separating boundary that $R(Z) = Z$. Let $Z_n := Z$. For every $0 \leq m < n$ we inductively define $Z_m := Z_{m+1} \setminus \mathrm{st}(C^{\downarrow}_{m+1})$ where $C_{m+1} \subset Z_{m+1}$ is some chamber with $\ell_{Z_{m+1}}(C_{m+1}) = 0$. Note that the existence of such chambers follows from the compactness of Z. In this situation Lemma 3.18 tells us that Z_m is a compact, σ-convex subcomplex of Z that satisfies $R(Z_m) = R(Z_{m+1})$ and $\mathrm{st}(C^{\downarrow}_{m+1}) \cap Z_{m+1} \subset \overline{C_{m+1}}$ for every $0 \leq m < n$. Note that the latter inclusion implies (2) and that the former equality gives us

$$R(Z) = R(Z_n) = R(Z_{n-1}) = \ldots = R(Z_1) = R(Z_0) = Z_0$$

which proves the claim. □

For short reference we note the following easy property of sectors.

Lemma 3.20. *Let $w \in \Sigma$ be a special vertex and let τ be a chamber in $\partial_\infty \Sigma$. Let A be a cell in the closed sector $\overline{K_w(\tau)}$. Then the projection chamber $\mathrm{pr}_A(\tau)$ lies in $\overline{K_w(\tau)}$.*

Proof. This follows directly from the fact that for every $\xi \in \tau$ and every $x \in \overline{K_w(\tau)}$ the ray $[x, \xi)$ stays in $\overline{K_w(\tau)}$. □

In particular Lemma 3.20 implies that for every panel P that lies in a wall of a sector $K_w(\tau)$, the projection chamber $\mathrm{pr}_P(\tau)$ lies in $K_w(\tau)$. This gives us the following.

Corollary 3.21. *Let $w \in \Sigma$ be a special vertex and let τ be a chamber in $\partial_\infty\Sigma$. Let $\Gamma = E_1|\dots|E_n$ be a τ-minimal gallery in Σ. If E_1 lies in $K_w(\tau)$ then the whole gallery Γ lies in $K_w(\tau)$.*

Lemma 3.22. *Let $w \in \Sigma$ be a special vertex. Let A be a cell in the sector complement $\Sigma \backslash K_w(\sigma)$. Then the projection $\mathrm{pr}_A(\sigma^{op})$ lies in $\Sigma \backslash K_w(\sigma)$.*

Proof. Suppose that $\mathrm{pr}_A(\sigma^{op}) \subset K_w(\sigma)$. Then A is a cell in $\overline{K_w(\sigma)}$ and by Lemma 3.20 $\mathrm{pr}_A(\sigma) \subset \overline{K_w(\sigma)}$. The convexity of the subspace $K_w(\sigma)$ implies that

$$\mathrm{conv}(\mathrm{pr}_A(\sigma), \mathrm{pr}_A(\sigma^{op})) = \mathrm{st}(A)$$

lies in $K_w(\sigma)$. Since $\mathrm{st}(A)$ is an open neighborhood of A it follows that A lies in the (open) sector $K_w(\sigma)$. This contradicts the choice of A. □

Proposition 3.23. *Let $w \in \Sigma$ be a special vertex. The closed sectors $\overline{K_w(\sigma)}$, $\overline{K_w(\sigma^{op})}$ and the complements $\Sigma \backslash K_w(\sigma)$ and $\Sigma \backslash K_w(\sigma^{op})$ are σ-convex.*

Proof. In view of Remark 3.14 it suffices to show that the complexes $\overline{K_w(\sigma)}$ and $\Sigma \backslash K_w(\sigma)$ are σ-convex. Since $\overline{K_w(\sigma)}$ is a convex subcomplex it follows that it is also σ-convex. Next we consider the complement $\Sigma \backslash K_w(\sigma)$. Let A and B be cells in $\Sigma \backslash K_w(\sigma)$ and suppose that there is a σ-minimal gallery $\Gamma = E_1|\dots|E_n$ from $E_1 = \mathrm{pr}_A(\sigma)$ to $E_n = \mathrm{pr}_B(\sigma^{op})$. By Lemma 3.22 the chamber $\mathrm{pr}_B(\sigma^{op})$ is contained in $\Sigma \backslash K_w(\sigma)$. Suppose that Γ contains a chamber E_{i_0} in $K_w(\sigma)$. Then the subgallery $\Gamma' := E_{i_0}|\dots|E_n$ is σ-minimal and hence by Lemma 3.21 stays in $K_w(\sigma)$. A contradiction to $E_n = \mathrm{pr}_B(\sigma^{op}) \subset \Sigma \backslash K_w(\sigma)$. □

Note that the property of being σ-convex behaves well under taking intersections.

Lemma 3.24. *The intersection of σ-convex complexes is σ-convex.* □

The non-separating boundary of the intersection of two subcomplexes can easily be described in terms of the subcomplexes as follows.

Lemma 3.25. *Let Y and Z be two subcomplexes of Σ. Then*

$$R(Y \cap Z) = Y \cap Z \cap (R(Y) \cup R(Z)).$$

Proof. This follows from the definition. □

Definition 3.26. Let $r \in \mathbb{R}$ be a real number and let $M(r)$ be the set of special vertices $w \in \Sigma$ of height $h(w) \geq r$. We define

$$U_h(r) = \bigcup_{w \in M(r)} \overline{K_w(\sigma^{op})}$$

to be the union of closed sectors corresponding to the vertices in $M(r)$ and the chamber σ^{op}. We will say that $U_h(r)$ is the *upper complex* associated to h and r. The *lower complex* associated to h and r, denoted by $L_h(r)$, is the complement of the interior of $U_h(r)$ in Σ, i.e.

$$L_h(r) = \Sigma \backslash \bigcup_{w \in M(r)} K_w(\sigma^{op}).$$

The following proposition summarizes some properties of the lower complex $L_h(r)$.

Proposition 3.27. *There is a constant $\varepsilon > 0$ such that for every $r \in \mathbb{R}$*

1. *$L_h(r)$ is σ-convex,*
2. *$h^{-1}((-\infty, r]) \subset L_h(r)$,*
3. *$L_h(r) \subset h^{-1}((-\infty, r+\varepsilon])$, and*
4. *$R(L_h(r)) \subset h^{-1}([r, r+\varepsilon])$.*

Proof. Note that we can write the lower complex as an intersection

$$L_h(r) = \Sigma \backslash \bigcup_{w \in M(r)} K_w(\sigma^{op}) = \bigcap_{w \in M(r)} \Sigma \backslash K_w(\sigma^{op})$$

of sector complements. Now the first claim follows directly from Lemma 3.24 and Corollary 3.23. To see the second claim recall that our choice of h implies that w is the lowest point of $K_w(\sigma^{op})$ and therefore

$$K_w(\sigma^{op}) \subset h^{-1}((r, \infty)) \text{ for every vertex } w \in M(r).$$

Thus the sublevelset $h^{-1}((-\infty, r])$ is completely contained in the sector complement $\Sigma \backslash K_w(\sigma^{op})$ for every vertex $w \in M(r)$. Therefore $h^{-1}((-\infty, r]) \subset L_h(r)$. Let $\varepsilon > 0$ be the constant from Lemma 3.12. To prove the third claim let $x \in L_h(r)$ be an arbitrary point. By Lemma 3.12 there is a special vertex $w \in \Sigma$ with $h(w) \geq h(x) - \varepsilon$ such that $x \in K_w(\sigma^{op})$. Suppose that $h(x) \geq r + \varepsilon$. In this case we have $w \in M(r)$ and therefore $x \notin L_h(r+\varepsilon)$ which is a contradiction. To see that the last claim is true, let A be a cell in $L_h(r)$. Suppose that $\mathrm{pr}_A(\sigma^{op})$ is not contained in $L_h(r)$. That is, there is a vertex $w \in M(r)$ such that $\mathrm{pr}_A(\sigma^{op}) \subset K_w(\sigma^{op})$. Then

$$A \subset \overline{K_w(\sigma^{op})} \subset h^{-1}([h(w), \infty)) \subset h^{-1}([r, \infty)).$$

On the other, hand the third claim gives us

$$A \subset L_h(r) \subset h^{-1}((-\infty, r+\varepsilon]).$$

□

Lemma 3.28. *There is a constant $\varepsilon > 0$ such that for every real number $r \in \mathbb{R}$ and every special vertex $w \in \Sigma$ the intersection*

$$\overline{K_w(\sigma^{op})} \cap L_h(r)$$

is σ-convex and its non-separating boundary satisfies

$$R(\overline{K_w(\sigma^{op})} \cap L_h(r)) \subset h^{-1}([r, r+\varepsilon]).$$

Proof. The complexes $\overline{K_w(\sigma^{op})}$ and $L_h(r)$ are σ-convex by Proposition 3.23 respectively by Proposition 3.27. Thus their intersection is σ-convex by Lemma 3.24. Let ε be as in Proposition 3.27. Lemma 3.25 tells us that

$$R(\overline{K_w(\sigma^{op})} \cap L_h(r)) = \overline{K_w(\sigma^{op})} \cap L_h(r) \cap \Big(R(\overline{K_w(\sigma^{op})}) \cup R(L_h(r))\Big).$$

Further, it follows from Lemma 3.20 that $R(\overline{K_w(\sigma^{op})}) = \emptyset$. Together with claim (4) of Proposition 3.27 this gives us

$$\begin{aligned} R(\overline{K_w(\sigma^{op})} \cap L_h(r)) &= \overline{K_w(\sigma^{op})} \cap L_h(r) \cap R(L_h(r)) \\ &\subset R(L_h(r)) \subset h^{-1}([r, r+\varepsilon]). \end{aligned}$$ □

For future reference we note the following feature of the non-separating boundary.

Lemma 3.29. *Let $r \in \mathbb{R}$ and let $w \in \Sigma$ be a special vertex. Let further*

$$P \subset R\Big(\overline{K_w(\sigma^{op})} \cap L_h(r)\Big) \cap K_w(\sigma^{op})$$

be a panel. The chamber $\mathrm{pr}_P(\sigma)$ lies in $K_w(\sigma^{op}) \cap L_h(r)$.

Proof. Suppose that $\mathrm{pr}_P(\sigma)$ does not lie in $K_w(\sigma^{op}) \cap L_h(r)$. Since by assumption P lies in the (open) sector $K_w(\sigma^{op})$ it follows that $\mathrm{pr}_P(\sigma)$ is contained in $K_w(\sigma^{op})$. Therefore $\mathrm{pr}_P(\sigma)$ is not contained in $L_h(r)$. Hence there has to be a vertex $w \in M(r)$ such that $\mathrm{pr}_P(\sigma)$ lies in $K_w(\sigma^{op})$. On the other hand Lemma 3.20 tells us that $\mathrm{pr}_P(\sigma^{op})$ lies in $K_w(\sigma^{op})$. Thus it follows that the star $\mathrm{st}(P)$ is contained in $K_w(\sigma^{op})$. Clearly this implies that P lies in $K_w(\sigma^{op})$ which shows that P is not contained in $L_h(r)$. Since this contradicts our choice of P we see that $\mathrm{pr}_P(\sigma)$ has to lie in $K_w(\sigma^{op}) \cap L_h(r)$. □

Proposition 3.30. *Let $r \in \mathbb{R}$ be a real number, $w \in \Sigma$ a special vertex, and let $Z = U_h(r) \cup K_w(\sigma^{op})$. Then there is a filtration*

$$U_h(r) = Z_1 \lneq Z_2 \lneq \ldots \lneq Z_n = Z$$

of Z by subcomplexes Z_i such that the following is satisfied for every $1 \leq m < n$.

1. *$Z_{m+1} = Z_m \cup \overline{C_{m+1}}$ for some chamber $C_{m+1} \subset Z_{m+1}$ with $\ell_{Z_{m+1}}(C_{m+1}) = 0$.*
2. *$\mathrm{st}(C^{\downarrow}_{m+1}) \cap Z_{m+1} \subset \overline{C_{m+1}}$.*

Proof. Let ε be as in Proposition 3.27. The third claim of Proposition 3.27 gives us $L_h(r) \subset h^{-1}((-\infty, r+\varepsilon])$. Since the intersection $\overline{K_w(\sigma^{op})} \cap h^{-1}((-\infty, r+\varepsilon])$ is compact by Lemma 3.10, it follows that there are only finitely many cells in $U_h(r) \cup \overline{K_w(\sigma^{op})}$ not lying in $U_h(r)$.

We saw in Lemma 3.28 that $Y = L_h(r) \cap \overline{K_w(\sigma^{op})}$ is σ-convex. Hence by Proposition 3.19 there is a filtration

$$R(Y) = Y_1 \lneq Y_2 \lneq \ldots \lneq Y_n = Y$$

such that for each $1 \leq m < n$

1. $Y_{m+1} = Y_m \cup \overline{D_{m+1}}$ for some chamber $D_{m+1} \subset Y_{m+1}$ with $\ell_{Y_{m+1}}(D_{m+1}) = 0$ and
2. $\mathrm{st}(D^{\downarrow}_{m+1}) \cap Y_{m+1} \subset \overline{D_{m+1}}$.

We claim that we obtain the desired filtration be setting $C_m = D_m$, $Z_1 = U_h(r)$, and $Z_{m+1} = Z_m \cup \overline{C_{m+1}}$ for every $1 \leq m < n$. To prove the first claim it suffices to show that $\ell_{Z_{m+1}}(C_{m+1}) = 0$. By construction we know that $\ell_{Y_{m+1}}(D_{m+1}) = 0$. Suppose that $\ell_{Z_{m+1}}(C_{m+1}) > 0$. Then there is a panel P of C_{m+1} such that $C_{m+1} \neq \mathrm{pr}_P(\sigma) \subset Z_{m+1}$. This shows that $C_{m+1} = \mathrm{pr}_P(\sigma^{op})$. Since $\ell_{Y_{m+1}}(D_{m+1}) = 0$ it follows that $\mathrm{pr}_P(\sigma)$ does not lie in Y_{m+1} and hence that $\mathrm{pr}_P(\sigma) \subset U_h(r)$. Thus there is a vertex $u \in M(r)$ such that $\mathrm{pr}_P(\sigma)$ is a chamber of $K_u(\sigma^{op})$. On the other hand Lemma 3.20 tells us that $\mathrm{pr}_P(\sigma^{op}) = C_{m+1}$ lies in $K_u(\sigma^{op})$. But this is a contradiction to our assumption that C_{m+1} is a chamber in $L_h(r)$.

To prove the second claim let $A \subset \mathrm{st}(C^{\downarrow}_{m+1}) \cap Z_{m+1}$ be a cell. Since

$$\mathrm{st}(C^{\downarrow}_{m+1}) \cap Y_{m+1} \subset \overline{C_{m+1}}$$

it suffices to consider the case where A does not lie in $\mathrm{st}(C^{\downarrow}_{m+1}) \cap Y_{m+1}$. Then A is a coface of $C^{\downarrow}_{m+1}$ that lies in $U_h(r)$. Hence there is a vertex $u \in M(r)$ such that $A \subset \overline{K_u(\sigma^{op})}$. In particular $C^{\downarrow}_{m+1} \subset \overline{K_u(\sigma^{op})}$ and Lemma 3.20 implies that

$$C_{m+1} = \mathrm{pr}_{C^{\downarrow}_{m+1}}(\sigma^{op}) \subset K_u(\sigma^{op}) \subset U_h(r).$$

But this is a contradiction to $C_{m+1} \subset Y \subset L_h(r)$. $\square$

4 THE POSITIVE DIRECTION IN TOP DIMENSION

In the previous chapter we considered filtrations of certain subcomplexes of Euclidean Coxeter complexes. In this chapter we will apply this filtration to filter subcomplexes of Euclidean buildings that appear as preimages of retractions from infinity. We start by fixing some notation. Let $X = \prod_{i=1}^{s} X_i$ be a product of thick, irreducible, Euclidean buildings X_i and let $d = \dim(X)$. We fix an apartment $\underline{\Sigma} = \prod_{i=1}^{s} \Sigma_i \subseteq X$ and a chamber $\sigma \subset \partial_\infty \underline{\Sigma}$. Recall that we view X as a CAT(0)-space and endow its boundary at infinity $\partial_\infty X$ with the structure of a spherical building. As in the last chapter we fix a special vertex $\underline{v} \in \underline{\Sigma}$ and think of $\underline{\Sigma}$ as a Euclidean vector space with origin $\underline{v}$. Further, we fix a non-trivial linear form $h \colon \underline{\Sigma} \to \mathbb{R}$ such that for every point $\xi \in \overline{\sigma}$ and every $x \in \underline{\Sigma}$ the function $h \circ [x, \xi) \colon [0, \infty) \to \mathbb{R}$ is strictly decreasing.

4.1 THE HEIGHT FUNCTION ON X

Our first goal is to extend h to a function on X. In order to do so we recall how to construct a retraction $\rho \colon X \to \underline{\Sigma}$ with respect to a chamber at infinity.

Definition 4.1. Let $\mathcal{A}_\sigma$ be the set of apartments of X that contain a subsector of $K_v(\sigma) \subset \underline{\Sigma}$. For each $\Sigma \in \mathcal{A}_\sigma$ let $f_\Sigma \colon \Sigma \to \underline{\Sigma}$ denote the isomorphism given by the building axiom $(B2)$.

The following definition makes sense since X is covered by the apartments in $\mathcal{A}_\sigma$ (see [2, Theorem 11.63.(1)]).

Definition 4.2. Let $\rho := \rho_{\sigma,\underline{\Sigma}} \colon X \to \underline{\Sigma}$ be the map given by $x \mapsto f_\Sigma(x)$ where Σ is any apartment in $\mathcal{A}_\sigma$ containing x.

It can be easily seen that $f_\Sigma(x) = f_{\Sigma'}(x)$ for every two apartments $\Sigma, \Sigma' \in \mathcal{A}_\sigma$ that contain x. Therefore ρ is well defined. In the following we will study the function $h \circ \rho \colon X \to \mathbb{R}$ and its superlevelsets in X. For convenience, we will just write h instead of $h \circ \rho$. This should not lead to confusions since the restriction of $h \circ \rho$ to $\underline{\Sigma}$ coincides with h. Note that the vector space structure on $\underline{\Sigma}$ allows us to consider the dual space $\underline{\Sigma}^* = \mathrm{Hom}(\underline{\Sigma}, \mathbb{R})$ for $\underline{\Sigma}$. The following definition provides us with a vector space of height functions of X.

Definition 4.3. We define $X^*_{\sigma,\underline{v}} = \{\alpha \circ \rho \colon \alpha \in \underline{\Sigma}^*\}$ to be the space of ρ-invariant linear forms on $\underline{\Sigma}$.

Remark 4.4. Note that indeed every function $f\colon X \to \mathbb{R}$ that is invariant under ρ, i.e. that satisfies $f \circ \rho = f$, and is linear on $\underline{\Sigma}$ lies in $X^*_{\sigma,\underline{v}}$. Note further that $X^*_{\sigma,\underline{v}}$ can be described as the space of ρ-invariant functions $X \to \mathbb{R}$ that are affine on some apartment $\Sigma \in \mathcal{A}_\sigma$ and vanish on $\underline{v}$.

For the rest of this chapter we will often just write $X^* := X^*_{\sigma,\underline{v}}$. Many of the spaces we are going to study involve the ρ-preimages of certain subsets of $\underline{\Sigma}$. In order to switch easily between subsets of $\underline{\Sigma}$ and their ρ-preimages in X we introduce the following notation.

Notation 4.5. For each subset $Z \subseteq \underline{\Sigma}$ we define $\widehat{Z} := \rho^{-1}(Z)$.

4.2 REMOVING RELATIVE STARS

Lemma 4.6. *Let A be a cell in X and let $C := \mathrm{pr}_A(\sigma)$ be the projection chamber.*

1. *For every chamber $D \subset \mathrm{st}(A)$ there is an apartment $\Sigma \in \mathcal{A}_\sigma$ such that $C, D \subset \Sigma$.*
2. *If $\Gamma = C_1|\dots|C_n \subset \mathrm{st}(A)$ is a minimal gallery terminating in C then the projection gallery $\rho(\Gamma)$ is also minimal in $\underline{\Sigma}$.*

Proof. Observe that the second claim follows from the first claim. Indeed, let $\Sigma \in \mathcal{A}_\sigma$ be an apartment containing C_1 and $C_n = C$. Since $\Sigma \in \mathcal{A}_\sigma$ it follows that the restriction $\rho_{|\Sigma}\colon \Sigma \to \underline{\Sigma}$ is an isomorphism. In particular, the minimal gallery Γ is mapped to the minimal gallery $\rho(\Gamma)$.

To prove the first claim let $\Sigma \in \mathcal{A}_\sigma$ be an apartment containing D. Let further $x \in A$ and $\xi \in \sigma$ be (interior) points. Since $\sigma \subset \partial_\infty\Sigma$ it follows that the ray $[x, \xi)$ is contained in Σ. On the other hand, there is an initial segment of the open geodesic ray (x, ξ) that lies in $\mathrm{pr}_A(\sigma) = C$. Thus we see that Σ contains a point of C and therefore we have $C \subset \Sigma$. □

Notation 4.7. Let $A \subset X$ be a cell of codimension ≥ 1. We say that two cofaces $B, C \subset \mathrm{st}(A)$ of A are *opposite to each other*, denoted by $B \,\mathrm{op}_{\mathrm{st}(A)}\, C$, if their corresponding simplices in $\mathrm{lk}(A)$ are opposite to each other.

The following consequence of Lemma 4.6 will help us to identify certain subcomplexes of X with complexes of the form $\mathrm{Opp}_\Delta(C)$ for some spherical buildings Δ and some chamber $C \subset \Delta$.

Corollary 4.8. *Let $A \subset X$ be a cell of codimension ≥ 1 and let $C = \mathrm{pr}_A(\sigma)$ denote the projection chamber of σ. We have $\mathrm{pr}_{\rho(A)}(\sigma) = \rho(C)$ and the equality*

$$\{D \subset \mathrm{st}(A)\colon D \,\mathrm{op}_{\mathrm{st}(A)}\, C\} = \{D \subset \mathrm{st}(A)\colon \rho(D) \,\mathrm{op}_{\mathrm{st}(A)}\, \rho(C)\}$$

is satisfied. This can be phrased more compactly by saying that the retraction commutes with taking opposite chambers of $\mathrm{pr}_A(\sigma)$.

The following definition extends Definition 3.7 to the case of Euclidean buildings.

Definition 4.9. For each chamber C in X we define the *upper face* $C^{\uparrow}$ of C to be the intersection of all panels $P < C$ such that $\mathrm{pr}_P(\sigma) = C$ (the face U in Lemma 3.6). The *lower face* $E^{\downarrow}$ of E is defined to be the intersection of all panels $P < C$ such that $\mathrm{pr}_P(\sigma) \neq C$.

We can rewrite Corollary 4.8 in terms of links rather than stars as follows.

Corollary 4.10. *Let* $Z \leq \Sigma$ *be a subcomplex. Suppose that* Z *contains a chamber* C *such that* $\mathrm{st}(C^{\downarrow}) \cap Z \subset \overline{C}$ *and let* A *be a cell in* $\rho^{-1}(C^{\downarrow})$. *Let* $a \subset \mathrm{lk}(A)$ *be the chamber induced by* $\mathrm{pr}_A(\sigma)$. *Then*

$$\mathrm{lk}_{\widehat{Z}}(A) = \mathrm{Opp}_{\mathrm{lk}_X(A)}(a).$$

The following definition will be crucial for showing that the system of superlevelsets $(X_{h\geq r})_{r\in\mathbb{R}}$ is essentially $(\dim(X) - 2)$-connected.

Definition 4.11. A spherical building Δ satisfies the *spherical opposition link property*, SOL-property, if the complexes $\mathrm{Opp}_{\mathrm{lk}(A)}(B)$ and $\mathrm{Opp}_{\Delta}(C)$ are spherical and non-contractible for every cell $A \subset \Delta$, every chamber $B \subset \mathrm{lk}(A)$, and every chamber $C \subset \Delta$.

Similarly we say that a Euclidean building Y satisfies the SOL-property if all of its links satisfy the SOL-property.

Lemma 4.12. *Assume that* X *satisfies the* SOL*-property. Let* $Z \subseteq \Sigma$ *be a subcomplex. Suppose that* Z *contains a chamber* C *such that* $\mathrm{st}(C^{\downarrow}) \cap Z \subset \overline{C}$ *and let* $Y := Z \setminus \mathrm{st}(C^{\downarrow})$. *The inclusion* $\iota\colon \widehat{Y} \to \widehat{Z}$ *induces monomorphisms*

$$\pi_k(\iota)\colon \pi_k(\widehat{Y}) \to \pi_k(\widehat{Z})$$

for every $0 \leq k \leq d-2$.

Proof. Let I be the set of cells in $\rho^{-1}(C^{\downarrow})$. Then $\widehat{Z}$ can be written as $\widehat{Z} = \widehat{Y} \cup \bigcup_{A\in I} \overline{\mathrm{st}_Z(A)}$. Note that we have

$$\overline{\mathrm{st}_{\widehat{Z}}(A)} \cap \overline{\mathrm{st}_{\widehat{Z}}(B)} \subset \widehat{Y}$$

for all cells $A, B \in I$ with $A \neq B$. Thus, in order to apply Lemma 2.20, we need to verify that

$$\overline{\mathrm{st}_{\widehat{Z}}(A)} \cap \widehat{Y}$$

is $(d-2)$-connected for every cell $A \in I$. Recall that by Remark 2.18 we have

$$\overline{\mathrm{st}_{\widehat{Z}}(A)} \cap \widehat{Y} = \partial\, \mathrm{st}_{\widehat{Z}}(A) \cong \partial A * \mathrm{lk}_{\widehat{Z}}(A).$$

Let c be the chamber in lk_Σ induced by $\mathrm{pr}_{C^\downarrow}(\sigma)$ and let $a \subset \mathrm{lk}_A$ be the chamber induced by $\mathrm{pr}_A(\sigma)$. On the other hand we have $\mathrm{lk}_{\widehat{Z}}(A) = \mathrm{Opp}_{\mathrm{lk}_X(A)}(a)$ by Corollary 4.10. The SOL-property of X tells us that the complex $\mathrm{Opp}_{\mathrm{lk}_X(A)}(\mathrm{pr}_A(\sigma))$ is $(\dim(\mathrm{lk}_X(A)) - 1)$-connected. Since $\partial(A)$ is homeomorphic to a sphere of dimension $\dim(A) - 1$ it follows from Lemma 2.21 that $\mathrm{Opp}_{\mathrm{lk}_X(A)}(\mathrm{pr}_A(\sigma)) * \partial(A)$ is k-connected for

$$k = (\dim(\mathrm{lk}(A)) - 1) + (\dim(A) - 2) + 2.$$

Now the claim follows from the simple observation

$$\dim(A) + \dim(\mathrm{lk}(A)) = d - 1. \qquad \square$$

Theorem 4.13. *Suppose that X satisfies the* SOL*-property. Let $r \in \mathbb{R}$ be a real number and let $v \in \Sigma$ be a special vertex. The canonical inclusion*

$$\iota\colon \widehat{U_h(r)} \to \widehat{U_h(r) \cup K_v(\sigma^{op})}$$

induces monomorphisms

$$\pi_k(\iota)\colon \pi_k(\widehat{U_h(r)}) \to \pi_k(\widehat{U_h(r) \cup K_v(\sigma^{op})})$$

for every $0 \leq k \leq d-2$.

Proof. By Proposition 3.30 there is a filtration

$$U_h(r) = Z_1 \lneq Z_2 \lneq \ldots \lneq Z_n = U_h(r) \cup K_v(\sigma^{op})$$

by subcomplexes Z_i such that the following is satisfied for each $1 \leq m < n$.

1. $Z_{m+1} = Z_m \cup \overline{C_{m+1}}$ for some chamber $C_{m+1} \subset Z_{m+1}$ with $\ell_{Z_{m+1}}(C_{m+1}) = 0$.
2. $\mathrm{st}(C^\downarrow_{m+1}) \cap Z_{m+1} \subset \overline{C_{m+1}}$.

In view of Lemma 4.12 this gives us a filtration

$$\widehat{U_h(r)} = \widehat{Z_1} \lneq \widehat{Z_2} \lneq \ldots \lneq \widehat{Z_n} = \widehat{U_h(r) \cup K_v(\sigma^{op})}$$

such that the each inclusion induces a monomorphism

$$\pi_k(\iota)\colon \pi_k(\widehat{Z_m}) \to \pi_k(\widehat{Z_{m+1}})$$

for every $0 \leq k \leq d-2$. Now the claim follows by composing these monomorphisms. $\square$

Corollary 4.14. *Suppose that X satisfies the* SOL*-property. Then $\widehat{U_h(r)}$ is $(d-2)$-connected for every $r \in \mathbb{R}$.*

Proof. Let $0 \leq k \leq d-2$ be an integer and let $f\colon S^k \to \widehat{U_h(r)}$ be a continuous function. Since X is a CAT(0)-space it is contractible. Hence there is a compact subspace $Z \subset X$ such that f can be contracted in Z. Then $\rho(Z) \subset \Sigma$ is also compact and Lemma 3.11 implies that there is a special vertex $v \in \Sigma$ such that $\rho(Z)$ is contained in the closed sector $\overline{K_v(\sigma^{op})}$. In particular we see that f is contractible in $\widehat{U_h(r)} \cup \widehat{\overline{K_v(\sigma^{op})}}$ and hence represents the trivial element in $\pi_k(\widehat{U_h(r)} \cup \widehat{\overline{K_v(\sigma^{op})}})$. On the other hand Theorem 4.13 says that the inclusion

$$\iota\colon \widehat{U_h(r)} \to \widehat{U_h(r)} \cup \widehat{\overline{K_v(\sigma^{op})}}$$

induces monomorphisms

$$\pi_k(\iota)\colon \pi_k(\widehat{U_h(r)}) \to \pi_k(\widehat{U_h(r)} \cup \widehat{\overline{K_v(\sigma^{op})}})$$

for each $0 \leq k \leq d-2$. Thus f represents the trivial element in $\pi_k(\widehat{U_h(r)})$ and therefore can be contracted in $\widehat{U_h(r)}$. Since f and k were chosen arbitrarily it follows that $\widehat{U_h(r)}$ is $(d-2)$-connected. □

Theorem 4.15. *Suppose that X satisfies the* SOL*-property. Then the system of superlevelsets $(X_{h\geq r})_{r\in\mathbb{R}}$ is essentially $(d-2)$-connected.*

Proof. Let $r \in \mathbb{R}$ be a real number. According to Proposition 3.27 there is a number $\varepsilon > 0$ such that we get a chain of inclusions

$$X_{h\leq r-\varepsilon} \to \widehat{L_h(r-\varepsilon)} \to X_{h\leq r}.$$

By considering the complements of these sets we see that the inclusion $\iota\colon X_{h\geq r} \to X_{h\geq r-\varepsilon}$ factorizes as

$$X_{h\geq r} \xrightarrow{\iota_1} \widehat{U_h(r-\varepsilon)} \xrightarrow{\iota_2} X_{h\geq r-\varepsilon}.$$

Now the claim follows since $\widehat{U(r+\varepsilon)}$ is $(d-2)$-connected by Corollary 4.14 and therefore the functoriality of π_k gives us

$$\pi_k(\iota) = \pi_k(\iota_2) \circ \pi_k(\iota_1) = 0 \text{ for every } 0 \leq k \leq d-2. \qquad \square$$

Remark 4.16. As far as I know this is the first time that a part of the computation of the Σ-invariants of a group benefits from the concept of *essential* n-connectedness. All the computations I am aware of show that the superlevelsets themselves are already n-connected.

5 THE NEGATIVE DIRECTION IN TOP DIMENSION

In the previous chapter we proved that certain systems of superlevelsets in an appropriate Euclidean building X are essentially $(\dim(X) - 2)$-connected. In this chapter we prove that these systems are not essentially $(\dim(X) - 1)$-connected.

As in the previous chapter we fix a product $X = \prod_{i=1}^{s} X_i$ of irreducible, Euclidean buildings X_i and let $d = \dim(X)$ denote its dimension. Further, we fix an apartment $\underline{\Sigma}$, a special vertex $\underline{v} \in \underline{\Sigma}$, and a chamber at infinity $\sigma \subset \partial_\infty \underline{\Sigma}$. The set of apartments in X that contain a subsector of $K_{\underline{v}}(\sigma)$ will be denoted by $\mathcal{A}_\sigma$.

5.1 THE ABSTRACT CONE

We start by constructing some cell complexes that are going to help us to transfer subcomplexes of $\partial_\infty X$ into subcomplexes of X.

Lemma 5.1. *For each chamber $\delta \subset \mathrm{Opp}_{\partial_\infty X}(\sigma)$ there is a unique apartment $\Sigma \in \mathcal{A}_\sigma$ such that $\delta \subset \partial_\infty \Sigma$.*

Proof. Since $\mathrm{Opp}_{\partial_\infty X}(\sigma)$ is a spherical building, the existence follows from the building axiom (B1). The uniqueness statement follows from the easy observation that every apartment is the convex hull of every pair of its sectors that correspond to opposite chambers. □

In view of Lemma 5.1 the following definition makes sense.

Definition 5.2. For every chamber $\delta \subset \mathrm{Opp}_{\partial_\infty X}(\sigma)$ let $\Sigma_\delta \in \mathcal{A}_\sigma$ denote the unique apartment with $\delta \subset \partial_\infty \Sigma_\delta$.

For the rest of this chapter we fix a compact subcomplex S of $\mathrm{Opp}_{\partial_\infty X}(\sigma)$ in which all maximal simplices are chambers.

Lemma 5.3. *There is a special vertex $v \in \bigcap_{\delta \in \mathrm{Ch}(S)} \Sigma_\delta$.*

Proof. This follows inductively from the observation that the intersection of two subsectors of $K_{\underline{v}}(\sigma)$ contains a common subsector. □

From now on we fix a special vertex v as in Lemma 5.3. Note that v can be regarded as the tip in the following construction.

Notation 5.4. For each subcomplex $Y \subset X$ let $\mathrm{Ch}(Y)$ denote the set of chambers in Y.

Definition 5.5. Let $K_{S,v} \subset X$ be the subcomplex consisting of the closed sectors in X that correspond to the chambers in S with tip in v, i.e.

$$K_{S,v} = \bigcup_{\delta \in \mathrm{Ch}(S)} \overline{K_v(\delta)}.$$

Recall that we use the notation $\widehat{Z} := \rho^{-1}(Z)$ where $\rho = \rho_{\sigma,\underline{\Sigma}}$ is the retraction from infinity defined in Definition 4.2.

Remark 5.6. Note that $K_{S,v}$ is a subcomplex of $\widehat{\overline{K_v(\sigma^{\mathrm{op}})}}$ where σ^{op} denotes the opposite chamber of σ in $\partial_\infty \underline{\Sigma}$. Note further that

$$\rho_{|\overline{K_v(\tau)}} \colon \overline{K_v(\tau)} \to \overline{K_v(\sigma^{\mathrm{op}})}$$

is an isomorphism for every chamber $\tau \subset S$.

In order to understand the structure of $K_{S,v}$ we introduce an auxiliary complex $\widetilde{K}_{S,v}$. This complex can be realized as a quotient space of the disjoint union $\coprod_{\delta \in \mathrm{Ch}(S)} \overline{K_v(\delta)}$ of closed sectors with tip in v that correspond to the chambers in S.

Definition 5.7. We say that two points $(p,\delta),(p',\delta') \in \coprod_{\delta \in \mathrm{Ch}(S)} \overline{K_v(\delta)}$ are equivalent, denoted by $(p,\delta) \sim (p',\delta')$, if $p = p' \in \partial\overline{K_v(P)}$ for some panel $P \subset \mathrm{Opp}_{\partial_\infty X}(\sigma)$. We define the *abstract cone* to be the quotient space

$$\widetilde{K}_{S,v} := \coprod_{\delta \in \mathrm{Ch}(S)} \overline{K_v(\delta)}/\sim .$$

Further we define the map $\pi \colon \widetilde{K}_{S,v} \to K_{S,v}, [(p,\delta)] \mapsto p$.

From now on we will often abbreviate the complexes $\widetilde{K}_{S,v}$ and $K_{S,v}$ by $\widetilde{K} = \widetilde{K}_{S,v}$ respectively $K = K_{S,v}$.

5.2 HOMOLOGY OF SUPERLEVELSETS

We keep the notations from the previous section. As always we regard $\underline{\Sigma}$ as a Euclidean vector space with origin $\underline{v}$. Recall from Definition 4.3 that $X^* := X^*_{\sigma,\underline{v}} = \{\alpha \circ \rho \colon \alpha \in \underline{\Sigma}^*\}$ denotes the space of ρ-invariant functions on X whose restrictions to $\underline{\Sigma}$ are linear. For the rest of this chapter we fix a function $h \in X^*$ such that $h \circ [x,\xi) \colon [0,\infty) \to \mathbb{R}$ is strictly decreasing for every $x \in \underline{\Sigma}$ and every $\xi \in \overline{\sigma}$.

Definition 5.8. Let A be a cell in K. The *branching number* of A, denoted by $b(A)$, is the number of chambers $\tau \subset S$ such that A is contained in $K_v(\tau)$.

Note for example that $b(v) = |\mathrm{Ch}(S)|$.

Remark 5.9. If E is a chamber in K then $b(E)$ is the number of chambers in the fiber $\pi^{-1}(E)$.

In order to prove that $(X_{h\geq r})_{r\in\mathbb{R}}$ is not essentially $(d-1)$-connected we will construct sequences of cycles in the (cellular) chain complex $C_{d-1}(X_{h\geq r};\mathbb{F}_2)$ of X. These cycles will appear as boundaries of d-chains in $C_d(X;\mathbb{F}_2)$ whose coefficients will depend on branching numbers.

Definition 5.10. For every $k \in \mathbb{N}_0$ and every k-chain

$$c = \sum_{A\subset X^{(k)}} \lambda_A \cdot A \in C_k(X;\mathbb{F}_2)$$

let $\operatorname{supp}(c)$ denote the set of all k-cells A with $\lambda_A = 1$.

A nice feature of working with affine cell complexes is that the attaching map is a homeomorphism for each closed cell. Thus the cellular boundary formula (see [20, Section 2.2]) gives us the following easy way of computing boundary maps.

Lemma 5.11. *For every $k \in \mathbb{N}_0$ the k-dimensional cellular boundary map of X is given by*

$$\partial_k \colon C_k(X;\mathbb{F}_2) \to C_{k-1}(X;\mathbb{F}_2), c \mapsto \sum_{A\subset X^{(k-1)}} \lambda_A \cdot A$$

where λ_A denotes the number of k-dimensional cofaces of A in $\operatorname{supp}(c)$.

Lemma 5.12. *For every chamber $\delta \subset S$ there is a special vertex $w \in K_v(\delta)$ such that $\overline{K_w(\delta)} \cap \Sigma_\tau = \emptyset$ for every chamber $\tau \in \mathrm{Ch}(S)\backslash\{\delta\}$.*

Proof. Let $\xi \in \delta$ be an arbitrary point. Since v lies in Σ_δ it follows that the ray $[v,\xi)$ is contained in Σ_δ. On the other hand we have $\xi \notin \tau$ for every $\tau \subset S$ with $\tau \neq \delta$. Since τ is the unique chamber in $\partial\Sigma_\tau$ that is opposite to σ we obtain $\xi \notin \partial\Sigma_\tau$. Hence for every $\tau \in \mathrm{Ch}(S)\backslash\{\delta\}$ there is a number $T_\tau > 0$ such that the point $[v,\xi)(T_\tau)$ is not contained in Σ_τ. Since S is finite we can choose T such that $p := [v,\xi)(T) \notin K_v(\tau)$ for every $\tau \subset S\backslash\{\delta\}$. Let $x \in \overline{K_p(\delta)}$ be an arbitrary point. Suppose that x is contained in Σ_τ for some $\tau \subset S$ with $\tau \neq \delta$. From the description of sectors given in Remark 3.9 it follows that $\overline{K_x(\sigma)}$ contains p. Since σ lies in $\partial_\infty\Sigma_\tau$ we see that the closed sector $\overline{K_x(\sigma)}$ is contained in Σ_τ and in particular that $p \in \Sigma_\tau$. But this contradicts our observation above. Since $x \in \overline{K_p(\delta)}$ was chosen arbitrarily it follows that $\overline{K_p(\delta)}$ is disjoint from Σ_τ for every $\tau \in \mathrm{Ch}(S)\backslash\{\delta\}$. Now the claim follows since $K_w(\delta) \subset K_p(\delta)$ for every special vertex $w \in K_p(\delta)$. □

Corollary 5.13. *For every chamber $\delta \subset S$ there is a special vertex $w \in K_v(\delta)$ such that $b(A) = 1$ for every cell $A \subset \overline{K_w(\delta)}$.*

In order to formulate the following definition we consider the function $\widetilde{K} \to \mathbb{R}, [(p,\tau)] \mapsto h(p)$ which we will also denote by h.

Definition 5.14. For every real number $r \in \mathbb{R}$ we define the space $K_r := K \cap X_{h\leq r}$. Analogously we define $\widetilde{K}_r$ to be the set of all points $p \in \widetilde{K}$ with $h(p) \leq r$.

Lemma 5.15. *The subspace $K_r \subset X$ is compact. In particular there are only finitely many chambers in K_r.*

Proof. Recall that $\overline{K_{\rho(v)}(\sigma^{\text{op}})} \cap X_{h\leq r} \subset \underline{\Sigma}$ is compact by Lemma 3.10. Since S is finite it follows from Remark 5.6 that K_r is the union of finitely many subcomplexes homeomorphic to $\overline{K_{\rho(v)}(\sigma^{\text{op}})} \cap X_{h\leq r}$. Thus we see that K_r is compact. □

In view of Lemma 5.15 the following definition makes sense.

Definition 5.16. For every real number $r \in \mathbb{R}$ we define the d-chain

$$c_r := \sum_{E\in \mathrm{Ch}(K_r)} b(E) \cdot E \in C_d(X; \mathbb{F}_2).$$

Remark 5.17. Note that the cycle c_r can also be described as the image of $\widetilde{c}_r := \sum_{E\in \mathrm{Ch}(\widetilde{K}_r)} E \in C_d(\widetilde{K}; \mathbb{F}_2)$ under the induced morphism

$$C_d(\pi)\colon C_d(\widetilde{K}; \mathbb{F}_2) \to C_d(K; \mathbb{F}_2).$$

Proposition 5.18. *There is a real number $R \in \mathbb{R}$ such that the boundary $\partial_d(c_r) \in C_{d-1}(X; \mathbb{F}_2)$ is non-zero for every $r \geq R$.*

Proof. Let $\delta \subset S$ be a chamber. By Corollary 5.13 there is a special vertex $w \in K_v(\delta)$ such that $b(A) = 1$ for every cell $A \subset K_w(\delta)$. Let $R \in \mathbb{R}$ be high enough such that K_R contains at least one chamber of $K_w(\delta)$ and let $r \geq R$. Since $K_w(\delta)$ is not bounded above with respect to h we can find a pair of adjacent chambers $E, F \subset K_w(\delta)$ such that $E \subset K_r$ but $F \not\subseteq K_r$. Let P be the common panel of E and F. Note that E is the unique chamber in K_r that lies in the star of P. In this case Lemma 5.11 tells us that the coefficient of P in the chain $\partial_d(c_r)$ is equal to 1 which proves the claim. □

5.3 ESSENTIAL NON-CONNECTEDNESS

We keep the definitions from the previous section. Further we make the assumption that the set of chambers $\mathrm{Ch}(S)$ consists of the support of a cycle

$$z := \sum_{\delta\in \mathrm{Ch}(S)} \delta \in Z_{d-1}(\mathrm{Opp}_{\partial_\infty X}(\sigma); \mathbb{F}_2).$$

Note that such a cycle exists if the complex $\partial_\infty X$ satisfies the SOL-property. Since z is a cycle it follows from Lemma 5.11 that every panel in S is a face of an even number of chambers of S. Note that, by construction of $\widetilde{K}$, this tells us that every panel $P \subset \widetilde{K}$ is a face of an even number of chambers in $\widetilde{K}$. This observation immediately implies the following.

Lemma 5.19. *Let $r \in \mathbb{R}$ be a real number and let $P \subset \widetilde{K}_r$ be a panel. If P is contained in an odd number of chambers in $\widetilde{K}_r$ then there is a chamber $E \subset \mathrm{st}(P)$ that contains a point p of height $h(p) > r$. In particular we see that the height of every point of P is bounded below by $r - \varepsilon$ where ε denotes the diameter of a chamber.* □

We now return to the chains c_r from the last section. In Proposition 5.18 we showed that $\partial(c_r) \in B_{d-1}(K;\mathbb{F}_2)$ is non-zero. The next proposition gives us a lower bound for the height of the panels in the support of $\partial(c_r)$.

Proposition 5.20. *There is a number $\varepsilon > 0$ such that for every $r \in \mathbb{R}$ the panels $P \in \mathrm{supp}(\partial(c_r))$ are contained in $X_{r \geq h \geq r-\varepsilon}$.*

Proof. Recall from Remark 5.17 that c_r is the image of the chain

$$\widetilde{c}_r = \sum_{E \in \mathrm{Ch}(\widetilde{K}_r)} E \in C_d(\widetilde{K};\mathbb{F}_2)$$

under the morphism $C_d(\pi)$. We consider the commutative diagram:

$$\begin{tikzcd} C_d(\widetilde{K}_r;\mathbb{F}_2) \arrow[r, "C_d(\pi)"] \arrow[d, "\partial_d"'] & C_d(K_r;\mathbb{F}_2) \arrow[d, "\partial_d"] \\ C_{d-1}(\widetilde{K}_r;\mathbb{F}_2) \arrow[r, "C_{d-1}(\pi)"'] & C_{d-1}(K_r;\mathbb{F}_2) \end{tikzcd}$$

From Lemma 5.19 we know that the height of all panels in $\mathrm{supp}(\partial_d(\widetilde{c}_r))$ is bounded below by $r - \varepsilon$. In particular, we see that the height of all panels in $\mathrm{supp}(C_{d-1}(\pi) \circ \partial_d(\widetilde{c}_r))$ is bounded below by $r - \varepsilon$. On the other hand, the above diagram tells us that

$$\partial_d(c_r) = \partial_d \circ C_d(\pi)(\widetilde{c}_r) = C_{d-1}(\pi) \circ \partial_d(\widetilde{c}_r)$$

which proves the claim. □

By combining Proposition 5.20 and Proposition 5.18 we get the following result.

Theorem 5.21. *For every real number $t > 0$ there is a level $s \in \mathbb{R}$ such that the inclusion*

$$\iota\colon X_{h \geq s+t} \to X_{h \geq s}$$

induces a non-trivial morphism

$$\widetilde{H}_{d-1}(\iota)\colon \widetilde{H}_{d-1}(X_{h \geq s+t};\mathbb{F}_2) \to \widetilde{H}_{d-1}(X_{h \geq s};\mathbb{F}_2)$$

Proof. From Corollary 5.13 it follows that there is a number $s \in \mathbb{R}$ and a chamber $E \subset X_{h \leq s}$ such that E is contained in the support of c_r for every $r \geq s$. Note further that on the one hand Proposition 5.18 tells us that the boundary $\partial_d(c_r)$ is non-zero for every sufficiently large $r \in \mathbb{R}$ and on the other hand Proposition 5.20 provides us with an $\varepsilon > 0$ such that every panel in the support of $\partial_d(c_r)$ is contained in $X_{r \geq h \geq r-\varepsilon}$. Thus we get a non-zero cycle $\partial_d(c_r) \in Z_{d-1}(X_{h \geq s+t}; \mathbb{F}_2)$ for some $r > s + t + \varepsilon$. Suppose that $\partial_d(c_r)$ is a boundary in $B_{d-1}(X_{h \geq s}; \mathbb{F}_2)$ and let $c \in C_d(X_{h \geq s}; \mathbb{F}_2)$ be a chain with $\partial_d(c) = \partial_d(c_r)$. Since the chamber $E \subset X_{h \leq s}$ is not contained in the support of $c \in C_d(X_{h \geq s}; \mathbb{F}_2)$ it follows that $c \neq c_r$. But this is a contradiction to the uniqueness statement on filling discs given in Lemma 2.22. □

In the following definition we recall a typical property of groups acting on buildings.

Definition 5.22. Let Δ be a building and let $\mathrm{Aut}(\Delta)$ denote the group of type preserving, cellular automorphisms of Δ. We say that a subgroup $G \leq \mathrm{Aut}(\Delta)$ acts *strongly transitively* on Δ if G acts transitively on the set of pairs (Σ, E) where Σ is an apartment of Δ and E is a chamber of Σ.

It turns out that Theorem 5.21 is exactly what is needed to show that the system $(X_{h \geq r})_{r \in \mathbb{R}}$ is not essentially $(d-1)$-connected. To see this we have to find isometries of X that act on the set of superlevelsets of X.

Lemma 5.23. *Suppose that* $\mathrm{Aut}(X)$ *acts strongly transitively on* X*. There is a non-zero constant* $a \in \mathbb{R}$ *and an isometry* $\alpha \in \mathrm{Aut}(X)$ *such that*

$$h(\alpha(x)) = h(x) + a, \forall x \in X.$$

Proof. Let $T\colon \underline{\Sigma} \to \underline{\Sigma}$ be a type preserving cellular translation such that $a := h(T(\underline{v})) - h(\underline{v}) \neq 0$. Let $E \subset \underline{\Sigma}$ be a chamber. Since the action of $\mathrm{Aut}(X)$ on X is strongly transitive there is an automorphism $\alpha\colon X \to X$ such that $\alpha(\underline{\Sigma}) = \underline{\Sigma}$ and $\alpha(E) = T(E)$. Since T, as a translation, is completely determined by its action on E it follows that $\alpha_{|\underline{\Sigma}} = T$. Recall that we denote the retraction associated to $\underline{\Sigma}$ and σ by ρ. We claim that $\rho \circ \alpha = \alpha \circ \rho$. Since X is covered by apartments in $\mathcal{A}_\sigma$ it suffices to prove this claim for every apartment in $\mathcal{A}_\sigma$. Thus we fix $\Sigma \in \mathcal{A}_\sigma$. Note that the restrictions of the maps $\rho \circ \alpha$ and $\alpha \circ \rho$ to Σ are isomorphisms to $\underline{\Sigma}$. In view of the obvious observation that a type preserving isomorphism between Coxeter groups is determined by the image of any chamber it is sufficient to find a chamber $C \subset \Sigma$ with $\rho \circ \alpha(C) = \alpha \circ \rho(C)$. By definition there is a sector $K_w(\sigma)$ lying in the intersection $\Sigma \cap \underline{\Sigma}$. In particular, we see that there is a chamber $C \subset \Sigma \cap \underline{\Sigma}$ and that

$$\rho \circ \alpha(C) = \rho \circ T(C) = T(C) = \alpha(C) = \alpha \circ \rho(C).$$

We claim that $h(\alpha(x)) - h(x) = a$ for every $x \in X$. Note that this is clear for $x \in \underline{\Sigma}$. Recall that h lies in X^*. Thus we have $h(\rho(x)) = h(x)$ for every $x \in X$ which gives us

$$\begin{aligned} h(\alpha(x)) - h(x) &= h(\rho \circ \alpha(x)) - h(\rho(x)) &&= h(\alpha \circ \rho(x)) - h(\rho(x)) \\ &= h(T(\rho(x))) - h(\rho(x)) &&= a \end{aligned}$$

for every $x \in X$. □

Corollary 5.24. *Suppose that* $\mathrm{Aut}(X)$ *acts strongly transitively on X. There is a non-zero constant $a \in \mathbb{R}$ such that for every $r \in \mathbb{R}$ and every $s < r$ there is a homeomorphisms of pairs $(X_{h \geq r}, X_{h \geq s}) \to (X_{h \geq r+a}, X_{h \geq s+a})$.*

We are now ready to prove the main theorem of this chapter. For easier reference we recall the assumptions on X we made along the way.

Theorem 5.25. *Let X be a d-dimensional Euclidean building, let $\sigma \subset \partial_\infty X$ be a chamber, and let $v \in X$ be a special vertex. Let further $h \in X^*_{\sigma,v}$ be such that $h \circ [x, \xi)$ is strictly decreasing for every $x \in X$ and every $\xi \in \overline{\sigma}$. Suppose that*

1. $\mathrm{Aut}(X)$ *acts strongly transitively on X and that*
2. *$\partial_\infty X$ satisfies the* SOL-*property.*

Then the system $(X_{h \geq r})_{r \in \mathbb{R}}$ is not essentially $(d-1)$-acyclic.

Proof. Suppose that $(X_{h \geq r})_{r \in \mathbb{R}}$ is essentially $(d-1)$-acyclic. Then there is some $r \in \mathbb{R}$ such that the morphism

$$\widetilde{H}_{d-1}(\iota) \colon \widetilde{H}_{d-1}(X_{h \geq r+R}; \mathbb{F}_2) \to \widetilde{H}_{d-1}(X_{h \geq r}; \mathbb{F}_2)$$

is trivial for some $R > 0$. In view of Corollary 5.24 this implies that there is a constant $a \neq 0$ such that the canonical morphisms

$$\widetilde{H}_{d-1}(X_{h \geq r+R+a \cdot k}; \mathbb{F}_2) \to \widetilde{H}_{d-1}(X_{h \geq r+a \cdot k}; \mathbb{F}_2)$$

are trivial for every $k \in \mathbb{Z}$. From Theorem 5.21 we know that there is some $s \in \mathbb{R}$ such that the morphism

$$\widetilde{H}_{d-1}(X_{h \geq s+t}; \mathbb{F}_2) \to \widetilde{H}_{d-1}(X_{h \geq s}; \mathbb{F}_2)$$

is non-trivial for $t := a + R$. By choosing $k \in \mathbb{Z}$ to be the smallest integer such that $r + a \cdot k \geq s$ we obtain the inequalities

$$s \leq r + a \cdot k \leq r + R + a \cdot k \leq s + t.$$

Note that this gives us the following commutative diagram where all maps are induced by inclusions.

$$\begin{tikzcd} \widetilde{H}_{d-1}(X_{h \geq s+t}; \mathbb{F}_2) \arrow[r] \arrow[d] & \widetilde{H}_{d-1}(X_{h \geq s}; \mathbb{F}_2) \\ \widetilde{H}_{d-1}(X_{h \geq r+R+a \cdot k}; \mathbb{F}_2) \arrow[r] & \widetilde{H}_{d-1}(X_{h \geq r+a \cdot k}; \mathbb{F}_2) \arrow[u] \end{tikzcd}$$

But this is a contradiction since the morphism at the bottom is trivial but the morphism at the top is not. □

6 CONVEX FUNCTIONS ON CAT(0)-SPACES

To prepare ourselves for some arguments appearing in the next chapter it will be useful to consider convex functions on CAT(0)-spaces in general. Our goal will be to find mild sufficient conditions under which convex functions on CAT(0)-spaces are continuous.

Definition 6.1. Let (X, d) be a CAT(0)-space. A function $f\colon X \to \mathbb{R}$ is called *convex* if for every two points $a, b \in X$ with $a \neq b$ and every point x on the geodesic segment $[a, b]$ the inequality

$$f(x) \leq \frac{d(x,a)}{d(a,b)} f(b) + \frac{d(x,b)}{d(a,b)} f(a)$$

holds.

Lemma 6.2. *Let (X, d) be a* CAT(0)*-space and let $f\colon X \to \mathbb{R}$ be a convex function. Let $a, b \in X$, $a \neq b$, let $0 < t < t' < d(a, b)$, and put $x = [a, b](t)$, $y = [a, b](t')$. Then the following inequalities are satisfied:*

$$\frac{f(x) - f(a)}{d(x,a)} \leq \frac{f(y) - f(x)}{d(y,x)} \leq \frac{f(b) - f(x)}{d(b,x)}.$$

Proof. Since y lies on the geodesic segment $[x, b]$ we may apply the convexity of f which gives us

$$f(y) \leq \frac{d(y,x)}{d(x,b)} f(b) + \frac{d(y,b)}{d(x,b)} f(x).$$

This gives us

$$\begin{aligned} f(y) - f(x) &\leq \frac{d(y,x)}{d(x,b)} f(b) + \frac{d(y,b)}{d(x,b)} f(x) - f(x) \\ &= \frac{d(y,x)}{d(x,b)} f(b) + \frac{d(y,b) - d(x,b)}{d(x,b)} f(x) \\ &= \frac{d(y,x)}{d(x,b)} f(b) - \frac{d(y,x)}{d(x,b)} f(x) \\ &= \frac{d(y,x)}{d(x,b)} (f(b) - f(x)). \end{aligned}$$

We therefore obtain the second inequality

$$\frac{f(y) - f(x)}{d(y,x)} \leq \frac{f(b) - f(x)}{d(b,x)}.$$

To obtain the first inequality we note that x lies on the geodesic segment $[a, y]$ and thus another application of the convexity of f gives us

$$f(x) \leq \frac{d(x,a)}{d(a,y)} f(y) + \frac{d(x,y)}{d(a,y)} f(a).$$

By rearranging this inequality we see that

$$f(y) \geq f(x) \cdot \frac{d(a,y)}{d(x,a)} - \frac{d(x,y)}{d(x,a)} f(a)$$

and hence by substracting $f(x)$ on both sides we obtain

$$\begin{aligned} f(y) - f(x) &\geq f(x) \cdot \frac{d(a,y)}{d(x,a)} - \frac{d(x,y)}{d(x,a)} f(a) - f(x) \\ &= f(x) \cdot \frac{d(a,y) - d(x,a)}{d(x,a)} - \frac{d(x,y)}{d(x,a)} f(a) \\ &= f(x) \cdot \frac{d(x,y)}{d(x,a)} - f(a) \frac{d(x,y)}{d(x,a)}. \end{aligned}$$

This implies

$$\frac{f(y) - f(x)}{d(x,y)} \geq \frac{f(x) - f(a)}{d(x,a)}. \qquad \square$$

In general there is no need for a convex function on a CAT(0)-space to be continuous. For example, it is easy to define linear (and hence convex) functions on infinite-dimensional topological vector spaces that are not continuous. The following definition aims to exclude such examples.

Definition 6.3. Let X be a topological space and let $f\colon X \to \mathbb{R}$ be an arbitrary (not necessarily continuous) function. The function f is called *locally bounded above* if for every point $x \in X$ there is a neighborhood U of x in X such that $f(U) \subset \mathbb{R}$ is bounded above.

Another type of convex non-continuous functions on CAT(0)-spaces can be defined on those CAT(0)-spaces that have some kind of a boundary. Consider for example the unit interval I. The function $f\colon I \to I$ that maps 1 to 1 and is constantly 0 elsewhere is convex but not continuous. In order to exclude such behavior we introduce the following property of geodesic metric spaces.

Definition 6.4. A geodesic metric space (X, d) is *locally uniformly extendible* if for every point $x \in X$ there are constants $\delta > 0$ and $\varepsilon > 0$ such that following property is satisfied. For every point $y \in B_\varepsilon(x)$ there is a geodesic segment $[a, b] \subset X$ containing some segment $[x, y]$ such that $d(a, x), d(b, y) \geq \delta$. In this case the constant δ will be called an *extendibility constant of x in X*.

Remark 6.5. Note that if δ is an extendibility constant of x in X then so is every number in the interval $(0, \delta]$.

It turns out that being locally bounded above for a function on a locally uniformly extendible CAT(0)-space is already enough to guarantee that the function is continuous.

Proposition 6.6. *Let (X, d) be a locally uniformly extendible* CAT(0)*-space. A convex function $f\colon X \to \mathbb{R}$ is continuous if and only if it is locally bounded above.*

Proof. It is clear that continuous functions are locally bounded. Thus let us assume that f is locally bounded above. Let $x \in X$ be an arbitrary point. Since f is locally bounded there are constants $\varepsilon > 0$ and $c \in \mathbb{R}$ such that $f(y) \leq c$ for every $y \in B_\varepsilon(x)$. By the above remark we can choose an extendibility constant $\delta \in (0, \frac{\varepsilon}{2})$ for $x \in X$.

Let $y \in B_{\frac{\varepsilon}{2}}(x)$ be a point with $y \neq x$. By the choice of δ it follows that there are two points $a, b \in B_\varepsilon(x)$ such that the geodesic segment $[a, b]$ contains the segment $[x, y]$ and that $d(a, x) = d(b, y) = \delta$. Thus an application of Lemma 6.2 gives us

$$\frac{f(x) - c}{\delta} \leq \frac{f(x) - f(a)}{d(x, a)} \leq \frac{f(y) - f(x)}{d(y, x)} \leq \frac{f(b) - f(x)}{d(b, x)} \leq \frac{c - f(x)}{\delta}.$$

Note that $c_1 := \frac{f(x)-c}{\delta}$ and $c_2 := \frac{c-f(x)}{\delta}$ do not depend on y and so it follows from the above inequality that

$$d(y, x) \cdot c_1 \leq f(y) - f(x) \leq d(x, y) \cdot c_2$$

which shows that f is continuous in x. □

The following application of Proposition 6.6 will be used in the next chapter.

Corollary 6.7. *Every convex function f on a locally compact Euclidean building X is continuous. In particular, convex functions on Euclidean vector spaces are continuous.*

Proof. Since (X, d) is a locally uniformly extendible CAT(0)-space, Proposition 6.6 tells us that it is sufficient to show that f is locally bounded above. Let $x \in X$ be a point. Since X is locally compact there is a compact neighborhood U of x such that U is covered by finitely many apartments $\{\Sigma_1, \ldots, \Sigma_k\}$. For every index $i \in \{1, \ldots, k\}$ let $U_i := \Sigma_i \cap U$. Since U_i is a bounded subset of the Euclidean space Σ_i we can find a finite set of points $\mathcal{V}_i \subset \Sigma_i$ such that U_i lies in the convex hull of $\mathcal{V}_i$. Thus every point in U_i can be written as a convex combination of the points in $\mathcal{V}_i$. From this it follows that the restriction of f to U_i is bounded by $c_i := \max_{v \in \mathcal{V}_i} f(v)$. Now the claim follows since the restriction of f to U is bounded above by $\max_{i \in \{1, \ldots, k\}} c_i$. □

7 PARABOLIC BUILDINGS

Let X be a Euclidean building and let $d = \dim(X) - 1$. In Chapter 4 and Chapter 5 we considered height functions $h \in X^*_{\sigma,v}$ for some chamber $\sigma \subset \partial_\infty X$ and some special vertex $v \in X$. We showed that, under certain conditions, the system of superleverlsets $(X_{h \geq r})_{r \in \mathbb{R}}$ is $(d-1)$-connected but not d-acyclic. One of these conditions was about h. We restricted ourselves to the case where the function $h \circ [x, \xi)$ is strictly decreasing for every point $\xi \in \overline{\sigma}$ and every $x \in X$. In this chapter we relax this condition by requiring that $h \circ [x, \xi) \colon [0, \infty)$ is non-increasing. We will see that none of the previous results hold in this generality. Nevertheless we will be able to apply the previous results on a building X^τ that is associated to some simplex $\tau \subset \partial_\infty X$. The idea of the construction of X^τ is to identify points in X that lie on a common geodesic ray $[x, \xi)$ where $\xi \in \overline{\tau}$ is such that $h \circ [x, \xi)$ is constant. In the case where τ is a panel this is well-known as the associated panel tree. In fact we will not so much speak about X^τ itself but rather about an isomorphic Euclidean building X_τ that appears as a convex subspace of X. Further it turned out to be convenient to introduce an auxiliary subbuilding $X^{\tau,\tau'}$ of X where $\tau' \subset \partial_\infty X$ is some simplex opposite to τ. Most of the time we restrict ourselves to the case where τ is a vertex and obtain the general case by iterating the construction. The buildings discussed in this chapter are described from an algebraic point of view in [22].

Definition 7.1. Let X be a Euclidean building and let $\xi \in \partial_\infty X$ be a vertex. Consider the set $\hat{X}^\xi$ of geodesic rays $[x, \xi) \subseteq X$ with the pseudo-distance

$$d([x, \xi), [y, \xi)) = \inf\{d(x', y') \mid x' \in [x, \xi), y' \in [y, \xi)\}.$$

The *parabolic building* X^ξ is the metric space obtained from $\hat{X}^\xi$ by identifying points of distance zero.

Remark 7.2. The space X^ξ can be constructed in much larger generality (see [17, Section 4.1]).

Definition 7.3. Let X be a Euclidean building and let $\xi, \xi' \in \partial_\infty X$ be opposite vertices. The *Levi building* associated to ξ and ξ', denoted by $X^{\xi,\xi'}$, is the set of geodesic lines connecting ξ and ξ' equipped with the distance function given by

$$d(\ell, m) = \inf\{d(x', y') \mid x' \in \ell, y' \in m\}.$$

The *extended Levi building* $\overline{X}^{\xi,\xi'}$ is a subspace of X. It is the union of all geodesic lines in $X^{\xi,\xi'}$.

We use the words *parabolic* and *Levi* since in the case where X is the Bruhat–Tits building associated to a Chavalley group $\mathcal{G}$, the spaces X^{ξ} and $X^{\xi,\xi'}$ correspond to the Bruhat–Tits buildings of the semisimple part of a parabolic respectively Levi subgroup of $\mathcal{G}$. Note that a priori it is not clear that X^{ξ} and $X^{\xi,\xi'}$ are Euclidean buildings. This will we shown in the first sections of this chapter. The above buildings are related by the following commutative diagram.

$$\begin{tikzcd}
\overline{X}^{\xi,\xi'} \arrow[r, two heads, "q"] \arrow[d, hook, "i"'] & X^{\xi,\xi'} \arrow[d, "j"] \\
X \arrow[r, two heads, "p"] & X^{\xi}
\end{tikzcd}$$

The maps p and q in the diagram are the quotient maps from Definition 7.1 and Definition 7.3 respectively. The map i denotes the inclusion and j is the map that takes a biinfinite line which is parametrized by a geodesic $c\colon \mathbb{R} \to X$ towards ξ and maps it to the class in X^{ξ} that is represented by the ray $c_{|[0,\xi)}$.

One of the goals in this chapter is to construct continuous sections for the maps p and q. Furthermore we will show that j is an isomorphism and that $\overline{X}^{\xi,\xi'}$ is a strong deformation retract of X.

7.1 APARTMENTS IN THE PARABOLIC BUILDING

Our first goal is to show that the spaces $X^{\xi}, X^{\xi,\xi'}$, and $\overline{X}^{\xi,\xi'}$ defined above can be naturally endowed with the structure of Euclidean buildings. For the rest of this chapter we fix a Euclidean building X, an apartment $\underline{\Sigma}$, a pair of opposite vertices $\xi, \xi' \in \partial_\infty \underline{\Sigma}$, and a chamber $\sigma \subset \partial_\infty \underline{\Sigma}$ that has ξ as a vertex. Further we fix a special vertex $\underline{v} \in \underline{\Sigma}$ which will allow us to view $\underline{\Sigma}$ as a vector space with origin $\underline{v}$. The full apartment system of X will be denoted by $\mathcal{A}$. The following types of apartments will be important for us.

Definition 7.4. An apartment $\Sigma \in \mathcal{A}$ is called *horizontal* if it contains the two opposite rays $[x,\xi)$ and $[x,\xi')$ for some (and hence every) point $x \in \Sigma$. The set of horizontal apartments of X will be denoted by $\mathcal{A}_{\mathrm{hor}}$. Analogously we say that a wall $H \subset X$ is *horizontal* if $[x,\xi)$ and $[x,\xi')$ are contained in H for some (and hence every) $x \in H$.

Note that our fixed apartment $\underline{\Sigma}$ is horizontal.

Definition 7.5. Let $\mathcal{H}$ denote the set of walls in $\underline{\Sigma}$ and let $\mathcal{H}_{\mathrm{hor}} \subset \mathcal{H}$ be the subset of horizontal walls. Let further $\mathcal{H}(\underline{v}) \subset \mathcal{H}$ denote the set of walls that contain $\underline{v}$ and let $\mathcal{H}_{\mathrm{hor}}(\underline{v}) = \mathcal{H}_{\mathrm{hor}} \cap \mathcal{H}(\underline{v})$.

By extending the geodesic germs in the link of the (special) vertex $\underline{v}$ to geodesic rays, we get an isomorphism $\mathrm{lk}_{\underline{\Sigma}}(\underline{v}) \to \partial_\infty \underline{\Sigma}$. In particular, it follows that the ray $[\underline{v}, \xi)$ contains a vertex $w \in \underline{\Sigma}$ that is incident to $\underline{v}$ via an edge e. Let e be fixed from now on. For the rest of this chapter let $E = \mathrm{pr}_{\underline{v}}(\sigma) \subset \mathrm{st}_{\underline{\Sigma}}(\underline{v})$. Note that by construction we further have $E \subset \mathrm{st}_{\underline{\Sigma}}(e)$. Let (W, S) be the Euclidean Coxeter system where S corresponds to the set of reflections in the hyperplanes of $\underline{\Sigma}$ that are spanned by the panels of E. Let further $(W_{\underline{v}}, S_{\underline{v}})$ be the spherical Coxeter system where $S_{\underline{v}}$ corresponds to the set of reflections in the hyperplanes $H \in \mathcal{H}(\underline{v})$ that are spanned by the panels of E. The theory of Euclidean Coxeter groups provides us with a decomposition $W = W_{\underline{v}} \ltimes L$ where L denotes the group of translations in W (see for example [2, Proposition 10.17.]).

Notation 7.6. The subgroup of W that is generated by the reflections s_H with $H \in \mathcal{H}_{\mathrm{hor}}$ will be denoted by W^{ξ}.

Our next goal is to construct a natural complex for W^{ξ} to act on. In order to define that complex we will consider the Busemann function $\beta\colon X \to \mathbb{R}$ that corresponds to ξ and $\underline{v}$ (see 2.9). We will frequently use the fact that $\underline{\Sigma} \cap \beta^{-1}(0)$ is a hyperplane in $\underline{\Sigma}$ that is orthogonal to the ray $[x, \xi)$ for every $x \in \underline{\Sigma} \cap \beta^{-1}(0)$. This is an easy exercise and can be found in [13, II.8.24.(1)]).

Lemma 7.7. *The group W^{ξ} stabilizes the sets $\underline{\Sigma} \cap \beta^{-1}(0)$ and $\mathcal{H}_{\mathrm{hor}}$. Further W^{ξ} fixes the point $\xi \in \partial_\infty \underline{\Sigma}$.*

Proof. Let H be a horizontal wall and let s_H be its corresponding reflection. By definition s_H fixes H pointwise. Let $x \in H$ be a point. Since H is horizontal it follows that the ray $[x, \xi)$ lies in H and gets fixed as well. This shows that s_H fixes ξ. Recall that $\underline{\Sigma} \cap \beta^{-1}(0)$ is a hyperplane that is orthogonal to the ray $[x, \xi)$. Since the isometry s_H fixes $[x, \xi)$ pointwise it has to stabilize its orthogonal complement $\underline{\Sigma} \cap \beta^{-1}(0)$ in order to respect the orthogonal decomposition. Since s_H fixes ξ it follows that $s_H([x, \xi)) = [s_H(x), \xi)$. This implies that the image of a horizontal hyperplane under the action of s_H is also horizontal. $\square$

Note that W^{ξ} does not have to be a parabolic subgroup of W. Yet we will see that W^{ξ} it is a Coxeter group in its own right and the natural space for it to act on is given by $\underline{\Sigma} \cap \beta^{-1}(0)$. For the rest of this chapter it will be convenient to write $d = \dim(X) - 1$.

Definition 7.8. Let $\mathcal{B} = \{\alpha_0, \ldots, \alpha_d\}$ be a set of linear forms in $\underline{\Sigma}^*$ such that

$$\overline{K_{\underline{v}}(\sigma)} = \{x \in \underline{\Sigma} \colon \alpha_k(x) \geq 0 \text{ for every } 0 \leq k \leq d\}.$$

Suppose further that $\alpha_0^{-1}(0)$ is the unique non-horizontal wall of $\overline{K_{\underline{v}}(\sigma)}$.

Clearly $\mathcal{B}$ is a basis of the dual space $\underline{\Sigma}^*$. It will be important for us to note that this is still the case if we replace α_0 by the restriction of β to $\underline{\Sigma}$. For convenience, we will denote this restriction by β as well.

Lemma 7.9. *The set of linear forms $\{\beta, \alpha_1, \dots, \alpha_n\}$ is a basis of $\underline{\Sigma}^*$.*

Proof. Note that the walls that bound $\overline{K_{\underline{v}}(\sigma)}$ are the kernels of the maps $\alpha_k \in \mathcal{B}$. Since α_0 is the unique linear form in $\mathcal{B}$ whose kernel is non-horizontal it follows that

$$\xi \in \partial_\infty(\bigcap_{k=1}^{d} \ker(\alpha_k)).$$

On the other hand, basic linear algebra tells us that $L := \bigcap_{k=1}^{d} \ker(\alpha_k)$ is a one dimensional linear subspace of $\underline{\Sigma}$ and therefore consists of the linear span of $[\underline{v}, \xi)$. Since $\underline{\Sigma} \cap \beta^{-1}(0)$ is a hyperplane in $\underline{\Sigma}$ that is orthogonal to $[\underline{v}, \xi)$ it further follows that

$$\ker(\beta) \cap \bigcap_{i=1}^{d} \ker(\alpha_i) = \{\underline{v}\}$$

and hence that $\{\beta, \alpha_1, \dots, \alpha_d\}$ forms a basis of $\underline{\Sigma}^*$. □

Definition 7.10. Let $\underline{\Sigma}_\xi = \underline{\Sigma} \cap \beta^{-1}(0)$ denote the zero level set of β in $\underline{\Sigma}$. Let further

$$\mathcal{H}_\xi = \{H \cap \underline{\Sigma}_\xi \colon H \in \mathcal{H}_{\text{hor}}\}$$

be the hyperplane arrangement that consists of the intersections of $\underline{\Sigma}_\xi$ with the horizontal hyperplanes in $\underline{\Sigma}$.

Remark 7.11. Note that it follows from Lemma 7.9 that $\mathcal{H}_\xi$ is indeed a system of hyperplanes in $\underline{\Sigma}$. From the construction we see that the map $\mathcal{H}_\xi \to \mathcal{H}_{\text{hor}}$ that maps a hyperplane H to the affine span of H and $[x, \xi)$ for some $x \in H$, is the inverse map of the restriction map $\mathcal{H}_{\text{hor}} \to \mathcal{H}_\xi, H \mapsto H \cap \beta^{-1}(0)$.

Proposition 7.12. *The space $\underline{\Sigma}_\xi$ can be endowed with the structure of a Euclidean Coxeter complex whose set of walls is given by the hyperplane arrangement $\mathcal{H}_\xi$. Its Coxeter group W_ξ is canonically isomorphic to W^ξ via the restriction map*

$$\phi \colon W^\xi \to W_\xi, f \mapsto f_{|\underline{\Sigma}_\xi}.$$

Proof. The local finiteness of $\mathcal{H}_\xi$ follows directly from the local finiteness of $\mathcal{H}$. Let $C \subset \underline{\Sigma}_\xi \setminus \bigcup_{H \in \mathcal{H}_\xi} H$ be a connected component. Since the kernels of the linear forms α_k are horizontal for every $1 \leq k \leq d$ it follows that C lies between two parallel walls of $\ker(\alpha_k)$ for every such k. Since these walls appear as preimages of certain real numbers it follows that the set $\alpha_k(C) \subset \mathbb{R}$ is bounded for every $1 \leq k \leq d$. Since C is a subset of $\underline{\Sigma}_\xi$ we further have $\beta(C) = 0$. In particular, we see that the image of C under the linear forms in $\{\beta, \alpha_1, \dots, \alpha_d\}$ is bounded. From Lemma 7.9 we know that this set is a basis of $\underline{\Sigma}^*$ which implies that C is a bounded subset of $\underline{\Sigma}_\xi$. It remains to show that $\mathcal{H}_\xi$ is stable under the action of W_ξ. To see this let $H_1, H_2 \in \mathcal{H}_\xi$ and let $s_1 = s_{H_1} \in W_\xi$. We have to show that $s_1(H_2)$ lies in $\mathcal{H}_\xi$. Let further $\overline{H_1}, \overline{H_2} \in \mathcal{H}_{\text{hor}}$ be hyperplanes with $H_i = \overline{H_i} \cap \underline{\Sigma}_\xi$ and let $\bar{s}_1 = s_{\overline{H_1}} \in W^\xi$. We claim that the restriction of $\bar{s}_1$ to $\underline{\Sigma}_\xi$ coincides with s_1. In view of Lemma 7.7 it follows that $\bar{s}_1$ restricts to an isometry $s_1' \colon \underline{\Sigma}_\xi \to \underline{\Sigma}_\xi$. From the construction it follows immediately that s_1 and s_1' are fixing H_1 pointwise. Thus in order to show that s_1' equals s_1 it suffices to prove that s_1' has order 2. Suppose that s_1' is not of order 2. Since the order of s_1' divides the order of s_1 it follows that s_1' is trivial. Thus it follows from Lemma 7.7 that $\bar{s}_1$ fixes the ray $[x, \xi)$ for every $x \in \underline{\Sigma}_\xi$. This shows that $\bar{s}_1$ is trivial on $\underline{\Sigma}$ and hence we get a contradiction to the assumption that $\bar{s}_1$ is a reflection. This argument shows in particular that the restriction map

$$\phi \colon W^\xi \to W_\xi, f \mapsto f_{|\underline{\Sigma}_\xi}$$

is injective. Next we prove that $\bar{s}_1(H_2)$ lies in $\mathcal{H}_\xi$. To see this we observe that

$$\bar{s}_1(H_2) = \bar{s}_1(\overline{H}_2 \cap \underline{\Sigma}_\xi) = \bar{s}_1(\overline{H}_2) \cap \bar{s}_1(\underline{\Sigma}_\xi) = \bar{s}_1(\overline{H}_2) \cap \underline{\Sigma}_\xi\,.$$

And thus the claim follows since $\bar{s}_1(\overline{H}_2)$ is a horizontal hyperplane by Lemma 7.7. Note that in particular we have proven that ϕ is also surjective. □

Corollary 7.13. *Let $\widetilde{\Phi}$ be the Coxeter diagram of W and let $\widetilde{\Psi}$ be the Coxeter diagram of W_ξ. The diagram Ψ is a subdiagram of Φ.*

Proof. Note that the map ϕ in Proposition 7.12 restricts to an isomorphism of stabilizers

$$\mathrm{st}_{W^\xi}(e) \to \mathrm{st}_{W_\xi}(\underline{v}).$$

Since every element in W that fixes e pointwise also fixes its linear span and hence the point ξ, it follows that the group $\mathrm{st}_{W^\xi}(e)$ coincides with the group $\mathrm{st}_W(e)$. From our construction it follows that $\mathrm{st}_W(e)$ is canonically isomorphic to the stabilizer of w in $W_{\underline{v}}$. The group $\mathrm{st}_{W_{\underline{v}}}(w)$ in turn can be described with the help of the well known-result on Coxeter groups that $\mathrm{st}_{W_{\underline{v}}}(w)$ is generated by all the standard generators of $W_{\underline{v}}$ that fix w pointwise (see e.g. [2, Theorem 1.104.]). In particular we see that $\mathrm{st}_{W_\xi}(\underline{v})$ is canonically isomorphic to a standard parabolic subgroup of the spherical Coxeter complex $W_{\underline{v}}$ and thus the claim follows. □

7.2 A SUBBUILDING OF X

Our next goal is to show that the extended Levi building $\overline{X}^{\xi,\xi'}$ is a subbuilding of X. We keep the previous notations from this chapter. The following lemma will be crucial in order to reach that goal.

Lemma 7.14. *For every point $x \in X$ there is a real number $t \in [0, \infty)$ such that $[x, \xi)(t) \in \overline{X}^{\xi,\xi'}$.*

Proof. From [2, Theorem 11.63.(1)] it follows that x is contained in an apartment Σ with the property $\sigma \subset \partial_\infty \Sigma$. Since $S := \partial_\infty \Sigma$ is an apartment in $\partial_\infty X$ Lemma 2.32 tells us that there is an apartment $S' \subset \partial_\infty X$ containing the star $\mathrm{st}_S(\xi)$ of ξ and the opposite vertex ξ' of ξ. By [2, Theorem 11.79] there is an apartment Σ' of X such that $\partial_\infty \Sigma' = S'$. In particular Σ' is a horizontal apartment and thus it suffices to show that $[x, \xi)(t) \in \Sigma'$ for some $t \geq 0$. For every chamber $\delta \subset \mathrm{st}_S(\xi)$ let $x_\delta \in \Sigma'$ be such that the sector $K_{x_\delta}(\delta)$ is contained in $\Sigma' \cap \Sigma$. It is easy to see that $[x, \xi)$ runs into the convex hull of these sectors. Now the claim follows since Σ' is a convex subcomplex of X. $\square$

By regarding X as a CAT(0)-space we may define the flow in X towards a point at infinity as follows.

Definition 7.15. Let $\eta \in \partial_\infty X$ be a point at infinity. The *flow in X towards η* is defined by

$$\Phi_\eta \colon X \times [0, \infty) \to X, (x, t) \mapsto [x, \eta)(t).$$

The following simple observation will help us to see that for every two points in $\overline{X}^{\xi,\xi'}$ there is a horizontal apartment containing both of them.

Lemma 7.16. *The complex $\overline{X}^{\xi,\xi'}$ is closed under taking the flows Φ_ξ and $\Phi_{\xi'}$. Further these flows commute on $\overline{X}^{\xi,\xi'}$ in the sense that for every $x \in \overline{X}^{\xi,\xi'}$ and every $t \geq 0$*

$$\Phi_{\xi'}(\Phi_\xi(x, t), t) = \Phi_\xi(\Phi_{\xi'}(x, t), t) = x.$$

Proof. Clearly this property holds on every line connecting ξ' and ξ. Hence the claim follows since $\overline{X}^{\xi,\xi'}$ is the union of such lines. $\square$

Corollary 7.17. *For every two points $x, y \in \overline{X}^{\xi,\xi'}$ and every real number $t \in [0, \infty)$ we have*

$$d(x, y) = d(\Phi_\xi(x, t), \Phi_\xi(y, t)) = d(\Phi_{\xi'}(x, t), \Phi_{\xi'}(y, t)).$$

Proof. In view of Lemma 7.16 we only have to show that the inequality $d(x, y) \geq d(\Phi_\eta(x, t), \Phi_\eta(y, t))$ holds for all $x, y \in X$, every point $\eta \in \partial_\infty X$, and every $t \geq 0$. Indeed, in this case Lemma 7.16 gives us

$$\begin{aligned} d(x, y) &\geq d(\Phi_\xi(x, t), \Phi_\xi(y, t)) \\ &\geq d(\Phi_{\xi'}(\Phi_\xi(x, t), t), \Phi_{\xi'}(\Phi_\xi(y, t), t)) \\ &= d(x, y). \end{aligned}$$

Since every CAT(0)-metric is convex (see [13, Proposition II.2.2]) it follows that the function $t \mapsto d([x,\eta)(t),[y,\eta)(t))$ is convex. Thus it remains to observe the simple fact that every convex, bounded function $[0,\infty) \to [0,\infty)$ is non-increasing. □

Definition 7.18. For every point $x \in \overline{X}^{\xi,\xi'}$ let $c_x \colon \mathbb{R} \to \overline{X}^{\xi,\xi'}$ be the unique geodesic that is determined by $c_x(0) = x$, $c_x(-\infty) = \xi'$, and $c_x(\infty) = \xi$.

Note that $c_x(\mathbb{R}) = q(x) \in X^{\xi,\xi'}$ where q is as in the introduction. In order to see that $\overline{X}^{\xi,\xi'}$ is a subbuilding of X we recall the following well-known fact about Euclidean buildings which can be found in [2, Theorem 11.53].

Theorem 7.19. *Let Y be a subset of X. Assume either that Y is convex or that Y has non-empty interior. If Y is isometric to a subset of $\mathbb{R}^d$, then Y is contained in an apartment.*

We are now ready to show that $\overline{X}^{\xi,\xi'}$ is a Euclidean building.

Lemma 7.20. *The space $\overline{X}^{\xi,\xi'}$ is a subbuilding of X. In particular, $\overline{X}^{\xi,\xi'}$ is a convex subspace. An apartment system for $\overline{X}^{\xi,\xi'}$ is given by $\mathcal{A}_{\mathrm{hor}}$.*

Proof. It suffices to show that every two points in $\overline{X}^{\xi,\xi'}$ lie in some horizontal apartment. Let $x, y \in \overline{X}^{\xi,\xi'}$ be arbitrary points and let $c_x, c_y \colon \mathbb{R} \to X$ be the corresponding isometric lines. Since c_x and c_y converge to the same ends at infinity it follows that the function

$$\mathbb{R} \to \mathbb{R}, \ t \mapsto d(c_x(t), c_y(t))$$

is bounded. Thus by the flat strip theorem (see [13, Theorem II.2.13]) it follows that the convex hull $\operatorname{conv}(c_x(\mathbb{R}), c_y(\mathbb{R}))$ is isometric to the strip $\mathbb{R} \times [0, D]$ for some $D \geq 0$. Hence $\operatorname{conv}(c_x(\mathbb{R}), c_y(\mathbb{R})) \subset X$ is a convex subspace which is isometric to a subset of the Euclidean space $\mathbb{R}^{d+1}$. In this case Theorem 7.19 implies that there is an apartment Σ of X containing $\operatorname{conv}(c_x(\mathbb{R}), c_y(\mathbb{R}))$. In particular we see that Σ is a horizontal apartment that contains x and y which proves the first claim. The second claim follows from the first since apartments are known to be convex subspaces of X. □

Lemma 7.21. *Let $H \subset \overline{X}^{\xi,\xi'}$ be a wall and let $P \subset H$ be a panel.*

1. *If H is a non-horizontal wall then there are exactly two chambers in $\overline{X}^{\xi,\xi'}$ that are incident to P.*
2. *If H is a horizontal wall then $\operatorname{st}_{\overline{X}^{\xi,\xi'}}(P) = \operatorname{st}_X(P)$.*

Proof. We start with the first claim. Thus let H be a non-horizontal wall. Let $P \subset H$ be a panel and let $p \in P$ be a point. The definition of horizontal apartments tells us that the two chambers C respectively D that contain some initial segments of (p, ξ) respectively (p, ξ') lie in every horizontal apartment that contains p. Since every chamber in $\mathrm{st}_{\overline{X}^{\xi,\xi'}}(P)$ is contained in a horizontal apartment (that contains p) by Lemma 7.20, we see that C and D are the only chambers in $\mathrm{st}_{\overline{X}^{\xi,\xi'}}(P)$.

Suppose now that H is a horizontal wall. Since $\overline{X}^{\xi,\xi'}$ is a building Theorem [2, 11.63.(1)] tells us that there is an apartment Σ in $\overline{X}^{\xi,\xi'}$ that contains P and such that $\sigma \subset \partial_\infty \Sigma$. Let R be the closed halfspace in Σ with $P \subset \partial R$ and $\sigma \subset \partial_\infty R$. Let D any chamber in $\mathrm{st}_X(P)$ and let $Z = R \cup D$. Since $\sigma \subset \partial_\infty \Sigma$ we see that the retraction $\rho_{\sigma,\Sigma}$ restricts to an isometric isomorphism $Z \to \rho_{\sigma,\Sigma}(Z)$. Thus Theorem 7.19 provides us with an apartment $\Sigma' \subset \overline{X}^{\xi,\xi'}$ that contains that contains Z which proves the claim. $\square$

Corollary 7.22. *The full apartment system of $\overline{X}^{\xi,\xi'}$ is given by $\mathcal{A}_{\mathrm{hor}}$.*

Proof. In view of Lemma 7.20 it remains to show that every apartment that is contained in $\overline{X}^{\xi,\xi'}$ is already a horizontal apartment. To see this let $\Sigma \subset \overline{X}^{\xi,\xi'}$ be an arbitrary apartment and let $x \in \Sigma$ be a point in some (open) chamber. We have to show that the rays $[x, \xi)$ and $[x, \xi')$ are contained in Σ. Let Σ' be a horizontal apartment that contains x. Recall that $\Sigma \cap \Sigma'$ is non-empty and convex since it is the intersection of convex spaces. In general one can show that every convex subcomplex of a Coxeter complex Π can be written as an intersection $\bigcap_{R \in \mathcal{R}(\Pi)} \overline{R}$, where $\mathcal{R}(\Pi)$ is the set of all half spaces in Π that are bounded by walls (see e.g. [2, Proposition 3.94]). Since Σ is a Euclidean space it is clear that in each parallel class of a half space in $\mathcal{R} = \mathcal{R}(\Sigma)$ there is at most one half space in that is necessary for the equality $\bigcap_{R \in \mathcal{R}} \overline{R} = \Sigma \cap \Sigma'$. Hence the finiteness of parallel classes of half spaces allows us to assume that $\mathcal{R}$ is finite and minimal, which we will do from now on. Let H be a wall that corresponds to some $R \in \mathcal{R}$, let $P \subset H \cap \Sigma \cap \Sigma'$ be a panel, and let $p \in P$ be an (interior) point of P. Suppose that H is non-horizontal. Then exactly one of the rays (p, ξ) and (p, ξ') has trivial intersection with $\Sigma \cap \Sigma'$. This implies that there are at least three chambers incident to P which contradicts Lemma 7.21. Hence $\Sigma \cap \Sigma'$ is the intersection of horizontal half spaces, i.e. half spaces corresponding to horizontal walls, and therefore it follows that the rays $[x, \xi)$ and $[x, \xi')$ are contained in these half spaces. Indeed, by assumption x lies in all of these half spaces and thus the claim follows from the trivial observation that neither $[x, \xi)$ nor $[x, \xi')$ can leave a half space that corresponds to a horizontal wall. $\square$

7.3 A NEW STRUCTURE FOR THE EXTENDED LEVI BUILDING

In 7.1 we saw that $\underline{\Sigma}_\xi = \underline{\Sigma} \cap \beta^{-1}(0)$ can be naturally endowed with the structure of a Euclidean Coxeter complex. Our next goal is to endow the space $X_\xi := \overline{X}^{\xi,\xi'} \cap \beta^{-1}(0)$ with the structure of a Euclidean building in a way that $\underline{\Sigma}_\xi$ is an apartment in X_ξ.

Definition 7.23. Let $\mathcal{H}_{\text{hor}}(X)$ denote the set of horizontal walls in X or, which is equivalent, the set of horizontal walls in $\overline{X}^{\xi,\xi'}$. Let further Λ be the union of all horizontal walls.

In order to define a building structure on X_ξ we have to introduce a new cell structure on X_ξ. The following lemma will help us to do so.

Lemma 7.24. *Let C be a component in $\overline{X}^{\xi,\xi'} \backslash \Lambda$. There is a horizontal apartment Σ of X such that C is a connected component in $\Sigma \backslash \Lambda$.*

Proof. Let $D \subset \overline{X}^{\xi,\xi'}$ be a chamber that is contained in C. By Lemma 7.20 there is a horizontal apartment Σ that contains D. We have to show that C is entirely contained in Σ. To see this let $F \subset C$ be a further chamber of $\overline{X}^{\xi,\xi'}$. Since C is open and connected it follows that there is a gallery Γ in C that starts in D and ends in F. Suppose that Γ leaves Σ at some point. In view of Lemma 7.21 there has to be a horizontal panel P that separates two chambers in Γ. By construction these two chambers lie in different components of $\overline{X}^{\xi,\xi'} \backslash \Lambda$. Hence we get a contradiction to our assumption that Γ is contained in C. □

Definition 7.25. Let $\mathcal{A}_\xi := \{\Sigma \cap X_\xi \colon \Sigma \in \mathcal{A}_{\text{hor}}\}$ denote the set of intersections of horizontal apartments with X_ξ.

Lemma 7.26. *Let $\overline{\Sigma} \in \mathcal{A}_{\text{hor}}$ be a horizontal apartment. The intersection $\Sigma := \overline{\Sigma} \cap X_\xi \in \mathcal{A}_\xi$ is a hyperplane in $\overline{\Sigma}$. It can be naturally endowed with the structure of a Euclidean Coxeter complex by defining the chambers to be the connected components in $\Sigma \backslash \Lambda$.*

Proof. Recall that we have proven the claim for $\underline{\Sigma}_\xi = \underline{\Sigma} \cap \beta^{-1}(0)$ in Proposition 7.12. Thus the general case follows from the observation that $\underline{\Sigma}$ was an arbitrary horizontal apartment. □

Thus we see that X_ξ is a cell complex that is covered by the Euclidean Coxeter complexes in $\mathcal{A}_\xi$. We are now ready to show that X_ξ is a Euclidean building.

Proposition 7.27. *The space X_ξ is a Euclidean building. Its full apartment system is given by $\mathcal{A}_\xi$.*

Proof. Since $\overline{X}^{\xi,\xi'}$ is covered by horizontal apartments it follows that X_ξ is covered by the apartments in $\mathcal{A}_\xi$. In order to check building axiom (B1) let A, B be cells in X_ξ and let a, b be points in A respectively

B. From Proposition 7.12 we know that there is a horizontal apartment $\overline{\Sigma}$ that contains a and b. Thus we get $a, b \in \Sigma := \overline{\Sigma} \cap \beta^{-1}(0) \in \mathcal{A}_\xi$. Since Σ is a subcomplex of X_ξ it follows that A and B are contained in Σ. To check building axiom (B2) let Σ_1, Σ_2 be two apartments in $\mathcal{A}_\xi$ and let $\overline{\Sigma}_1$ and $\overline{\Sigma}_2$ be the corresponding horizontal apartments. Suppose that Σ_1 and Σ_2 contain a chamber c of Σ_1 and let $C \subset \overline{\Sigma}_1 \backslash \Lambda$ be the component that contains c. From Lemma 7.24 it follows that $\overline{\Sigma}_1$ and $\overline{\Sigma}_2$ contain C. Since $\overline{X}^{\xi,\xi'}$ is a building it follows that there is an isomorphism $f \colon \overline{\Sigma}_1 \to \overline{\Sigma}_2$ that fixes the intersection $\overline{\Sigma}_1 \cap \overline{\Sigma}_2$ pointwise (see [2, Remark 4.5]). Since f fixes C pointwise it follows that f restricts to a map $\Sigma_1 \to \Sigma_2$ that maps horizontal walls to horizontal walls. In particular we see that the restriction of f to Σ_1 is an isomorphism that fixes the intersection $\Sigma_1 \cap \Sigma_2$ pointwise. In view of [2, Remark 4.4.] it follows that X_ξ is a Euclidean building. Note that the proof in particular implies that X_ξ is a convex subspace of $\overline{X}^{\xi,\xi'}$. Indeed, let x, y be a arbitrary points in $\overline{X}^{\xi,\xi'}$ and let $\Sigma \in \mathcal{A}_\xi$ be an apartment containing them. Then Σ is a hyperplane of an (horizontal) apartment in $\overline{X}^{\xi,\xi'}$ and thus the geodesic segment $[x, y]$ is contained in Σ. To prove the second claim let $\Sigma \subset X_\xi$ be an arbitrary apartment. Since $\Sigma \subset X_\xi \subset \overline{X}^{\xi,\xi'}$ is a chain of convex subspaces it follows that Σ is convex in $\overline{X}^{\xi,\xi'}$. Theorem 7.19 therefore implies that there is an apartment $\widetilde{\Sigma} \subset \overline{X}^{\xi,\xi'}$ that contains Σ. From Corollary 7.22 we know that $\widetilde{\Sigma}$ is horizontal and thus $\Sigma = \widetilde{\Sigma} \cap X_\xi \in \mathcal{A}_\xi$. □

Since our buildings are not necessarily irreducible it usually happens that some panels are faces of more chambers than others. Thus we can only estimate on the thickness of X_ξ.

Remark 7.28. The thickness of X_ξ can be bounded below by the thickness of X, i.e. $\mathrm{th}(X_\xi) \geq \mathrm{th}(X)$. If X is locally finite then so is X_ξ.

Since the cell structure on X_ξ is obtained by intersections of horizontal walls with X_ξ we get the following consequence of Lemma 7.21.

Corollary 7.29. *For every cell $A \subset X_\xi$ there is a cell $B \subset X$ such that $\mathrm{lk}_{X_\xi}(A) \cong \mathrm{lk}_X(B)$. In particular we see that if $\widetilde{\Phi}$ is the Coxeter diagram of X and $\widetilde{\Psi}$ is the Coxeter diagram of X_ξ then Ψ is a subdiagram of Φ.*

Remark 7.30. By analogue arguments we can also see that $\partial_\infty X_\xi$ is isomorphic to $\mathrm{lk}_{\partial_\infty X}(\xi)$.

8 THE GEOMETRIC MAIN RESULT

In this chapter we reduce the question of deciding whether $(X_{h\geq s})_{r\in\mathbb{R}}$ is essentially k-connected for some $k\in\mathbb{N}_0$ to the question whether some corresponding system of superlevelsets $(X^{\xi}_{h\geq r})_{r\in\mathbb{R}}$ in X^{ξ} is essentially k-connected.

As in the last chapter we fix a Euclidean building X, an apartment $\underline{\Sigma}$, a pair of opposite vertices $\xi,\xi'\in\partial_\infty\underline{\Sigma}$, and a chamber $\sigma\subset\partial_\infty\underline{\Sigma}$ that has ξ as a vertex. Further we fix a special vertex $\underline{v}\in\underline{\Sigma}$ which will allow us to view $\underline{\Sigma}$ as a vector space with origin $\underline{v}$. The full apartment system of X will be denoted by $\mathcal{A}$.

In this chapter we further fix a height function $h\in X^*_{\sigma,\underline{v}}$ with $\sigma\subset\partial_\infty\underline{\Sigma}_{h\leq 0}$ (see 4.3 for the definition of $X^*_{\sigma,\underline{v}}$). Suppose further that $\xi\in\partial_\infty\underline{\Sigma}_{h=0}$, or equivalently, that $h\circ[x,\xi)$ is constant for every $x\in X$.

In view of Lemma 7.14 the following definition makes sense.

Definition 8.1. Let $T\colon X\to\mathbb{R}$, $x\mapsto T(x)$ be the function where $T(x)$ is the smallest real number such that $[x,\xi)(T(x))\in\overline{X}^{\xi,\xi'}$. Let further $M\colon X\to\overline{X}^{\xi,\xi'}$ be the map given by $x\mapsto[x,\xi)(T(x))$.

One can think of M as a merging function. Note that the restriction of M to $\overline{X}^{\xi,\xi'}$ is the identity.

Lemma 8.2. *The functions $T\colon X\to\mathbb{R}$ and $M\colon X\to\overline{X}^{\xi,\xi'}$ are continuous.*

Proof. Since X is a cell complex that is covered by apartments Σ with $\sigma\subset\partial_\infty\Sigma$ it suffices to show that the restrictions of T and M to every such apartment are continuous. Thus let Σ be an apartment with $\sigma\subset X$. Note that this condition implies in particular that Σ is closed under the flow towards σ. Since apartments are convex, Lemma 7.20 implies that the intersection $\Sigma\cap\overline{X}_{\xi,\xi'}$ is convex. By combining these facts we see that the restriction $T_{|\Sigma}\colon\Sigma\to\mathbb{R}$ is a convex function. Thus the continuity of T follows from Corollary 6.7 which tells us that convex functions on finite dimensional Euclidean vector spaces are continuous. Next we prove the continuity of the merging function M. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in Σ converging to some point $x\in\Sigma$. We have to show that $[x_n,\xi)(T(x_n))$ converges to $[x,\xi)(T(x))$. By the triangle inequality we have

$$d([x_n,\xi)(T(x_n)),[x,\xi)(T(x)))$$

$$\leq d([x_n,\xi)(T(x_n)),[x,\xi)(T(x_n)))+d([x,\xi)(T(x_n)),[x,\xi)(T(x))).$$

Thus it remains to observe that

$$d([x_n,\xi)(T(x_n)),[x,\xi)(T(x_n)))\leq d(x_n,x)\to 0, n\to\infty$$

and that the continuity of T gives us

$$d([x,\xi)(T(x_n)),[x,\xi)(T(x))) = |T(x_n) - T(x)| \to 0, n \to \infty. \qquad \square$$

As a composition of continuous functions we obtain a homotopy from id_X to M.

Corollary 8.3. *The function*

$$H\colon X \times [0,1] \to X,\ (x,t) \mapsto [x,\xi)(t \cdot T(x))$$

is a homotopy from id_X *to* M *relative to* $\overline{X}^{\xi,\xi'}$. *Furthermore by restricting* H *to the* h*-superlevel sets of* X *we see that for every two real numbers* $s \leq t$ *the pair* $(\overline{X}^{\xi,\xi'}_{h\geq s}, \overline{X}^{\xi,\xi'}_{h\geq t})$ *is a strong deformation retraction of the pair* $(X_{h\geq s}, X_{h\geq t})$.

We close this section by translating Corollary 8.3 to the analogous statements for the buildings X_ξ and X^ξ. In order to do so we define a height function on X^ξ by $[[x,\xi)] \mapsto h(x)$ which we denote by h^ξ. Note that h^ξ is well defined since h was defined to be constant on every ray $[x,\xi)$.

Corollary 8.4. *The subspace* X_ξ *is a strong deformation retract of* $\overline{X}^{\xi,\xi'}$ *and hence of* X. *Furthermore for every two real numbers* $s \leq t$ *the pair* $((X_\xi)_{h\geq s}, (X_\xi)_{h\geq t})$ *is a strong deformation retract of the pair* $(\overline{X}^{\xi,\xi'}_{h\geq s}, \overline{X}^{\xi,\xi'}_{h\geq t})$ *and hence of the pair* $(X_{h\geq s}, X_{h\geq t})$. *Further we have a homotopy equivalence of pairs*

$$(X_{h\geq s}, X_{h\geq t}) \simeq (X^\xi_{h^\xi\geq s}, X^\xi_{h^\xi\geq t}).$$

Proof. The first claims follow from the simple observation that for every horizontal apartment Σ the subspace $\Sigma \cap \beta^{-1}(0)$ is a strong deformation retract. To see that last claim it now suffices to show that the quotient map $p\colon X \to X^\xi$, which was defined in the introduction, restricts to an isomorphism on X_ξ. This in turn is a direct consequence of Lemma 7.14. $\square$

Note in particular that for every $k \in \mathbb{N}_0$, the system $(X_{h\geq r})_{r\in\mathbb{R}}$ is essentially k-connected if and only if the system $(X^\xi_{h^\xi\geq r})_{r\in\mathbb{R}}$ is essentially k-connected, or which is equivalent, if the system $((X_\xi)_{h\geq r})_{r\in\mathbb{R}}$ is essentially k-connected.

8.1 RETRACTIONS IN PARABOLIC BUILDINGS

We keep the previous notations from this chapter. Further, we fix the retraction $\rho := \rho_{\underline{\Sigma},\sigma}\colon X \to \underline{\Sigma}$. Our goal in this section is to study retraction-invariant height functions on X_ξ and to see how these functions are related to those in $X^* = X^*_{\sigma,\underline{v}}$.

Recall from Definition 7.8 that $\mathcal{B} = \{\alpha_0, \ldots, \alpha_d\}$ is a set of linear forms on $\underline{\Sigma}$ such that

$$\overline{K_{\underline{\mathrm{v}}}(\sigma)} = \{x \in \underline{\Sigma}\colon \alpha_k(x) \geq 0, \text{ for every } 0 \leq k \leq d\}.$$

Recall further that $\xi_k \in \sigma^{(0)}$ denotes the vertex that is not contained in $\partial_\infty(\ker(\alpha_k))$ and that $\xi_0 = \xi$.

Definition 8.5. For every point $p \in \underline{\Sigma}$ we define the cone

$$K_p^{\xi,\xi'}(\sigma) = \{x \in \underline{\Sigma}\colon \alpha_k(x) \geq \alpha_k(p) \text{ for every } 1 \leq k \leq d\} \subset \underline{\Sigma}.$$

Furthermore, we define $K_{p,\xi}(\sigma) = K_p^{\xi,\xi'}(\sigma) \cap \underline{\Sigma}_\xi$ if p lies in $\underline{\Sigma}_\xi$.

The complexes $K_p^{\xi,\xi'}(\sigma)$ and $K_{p,\xi}(\sigma)$ defined above can be seen as analogous of $K_p(\sigma)$ for the present setting.

Lemma 8.6. *The space $K_{\underline{\mathrm{v}},\xi}(\sigma)$ is a sector in $\underline{\Sigma}_\xi$.*

Proof. Recall that $E = \mathrm{pr}_{\underline{\mathrm{v}}}(\sigma) \subset \mathrm{st}_{\underline{\Sigma}}(\underline{\mathrm{v}})$. Let further C be the component in $\underline{\Sigma} \setminus \Lambda$ that contains E and let $E_\xi = C \cap \underline{\Sigma}_\xi$ be the corresponding chamber in $\underline{\Sigma}_\xi$. Since $\ker(\alpha_i)$ is a horizontal wall that is spanned by a panel of E for every $1 \leq i \leq n$ it follows that the wall $\underline{\Sigma}_\xi \cap \ker(\alpha_i)$ is spanned by a panel of E_ξ for every $1 \leq i \leq n$. By considering the dimension of X_ξ it follows that every panel of E_ξ that contains $\underline{\mathrm{v}}$ spans a wall of the form $\underline{\Sigma}_\xi \cap \ker(\alpha_i)$ with $1 \leq i \leq n$. Now the claim follows since $K_{\underline{\mathrm{v}},\xi}(\sigma)$ coincides with the sector associated to $\underline{\mathrm{v}}$ and E_ξ. $\square$

It follows from Lemma 8.6 that $K_{p,\xi}(\sigma) \subset \underline{\Sigma}_\xi$ is a sector for every $p \in \underline{\Sigma}_\xi$. In particular, we can see that the following definition makes sense.

Definition 8.7. We consider the chamber $\sigma_\xi := \partial_\infty K_{\underline{\mathrm{v}},\xi}(\sigma) \subset \partial_\infty \underline{\Sigma}_\xi$ and let $\mathcal{A}_\xi^\sigma$ denote the set of apartments $\Sigma \in \mathcal{A}_\xi$ with $\sigma \subset \partial_\infty \Sigma$. The retraction from infinity corresponding to $\underline{\Sigma}_\xi$ and σ_ξ will be denoted by $\rho_\xi := \rho_{\underline{\Sigma}_\xi,\sigma_\xi}$.

Remark 8.8. Recall that $\overline{X}^{\xi,\xi'}$ is a building with $\sigma \subset \partial_\infty \overline{X}^{\xi,\xi'}$ and that the full apartment system of $\overline{X}^{\xi,\xi'}$ is given by $\mathcal{A}_{\mathrm{hor}}$. Thus it follows from [2, Theorem 11.63.] that $\overline{X}^{\xi,\xi'}$ is covered by the union of horizontal apartments that contain σ in their boundary. In particular, we see that X_ξ is covered by the apartments in $\mathcal{A}_\xi^\sigma$.

Lemma 8.9. *Let $\Sigma \subset \overline{X}^{\xi,\xi'}$ be a (horizontal) apartment with $\sigma \subset \partial_\infty \Sigma$. There is a point $p \in \Sigma \cap \underline{\Sigma}_\xi$ such that $K_p(\sigma_\xi)$ is contained in $\Sigma \cap \underline{\Sigma}_\xi$. In particular, we have $\sigma_\xi \subset \partial_\infty(\Sigma \cap \underline{\Sigma}_\xi)$.*

Proof. By definition σ lies in the boundaries of $\underline{\Sigma}$ and Σ. Thus we can find a sector $K_x(\sigma)$ in $\underline{\Sigma} \cap \Sigma$. It can be easily seen that $K_x(\sigma)$ is the convex hull of the rays $[x, \xi_i)$ where $0 \le i \le n$ and hence can be written as

$$K_x(\sigma) = \{\sum_{i=0}^{n} [x, \xi_i)(t_i) \colon t_i \ge 0 \text{ for } 0 \le i \le n\}.$$

Since $\underline{\Sigma}$ and Σ are horizontal it follows that the rays $[y, \xi)$ and $[y, \xi')$ lie in $\underline{\Sigma} \cap \Sigma$ for every $y \in \underline{\Sigma} \cap \Sigma$. The observation above therefore implies that

$$Z := \{\sum_{k=0}^{n} [x, \xi_k)(t_k) \colon t_0 \in \mathbb{R} \text{ and } t_i \ge 0 \text{ for } 1 \le i \le n\}$$

is contained in $\underline{\Sigma} \cap \Sigma$. Let $p \in Z$ let $c_p \colon \mathbb{R} \to X$ be the ray given Definition 7.18. The construction of the linear forms α_i tells us that $\alpha_i \circ c_p$ is constant for every $1 \le i \le n$ and can take arbitrary values for $i = 0$. This shows that Z coincides with $K_x^{\xi,\xi'}(\sigma)$. In particular we see that $K_{p,\xi}(\sigma)$ is contained in Z and hence in $\underline{\Sigma} \cap \Sigma$. Now the claim follows by choosing p to be in $\underline{\Sigma}_\xi$. □

Lemma 8.10. *The retraction $\rho_\xi \colon X_\xi \to \underline{\Sigma}_\xi$ coincides with the restriction of ρ to X_ξ.*

Proof. Let $\widetilde{\Sigma} \in \mathcal{A}_{\mathrm{hor}}$ be an apartment with $\sigma \subset \widetilde{\Sigma}$ and let $\Sigma = \widetilde{\Sigma} \cap X_\xi$. From Lemma 8.9 we know that Σ contains the sector $K_p(\sigma_\xi) \subset \underline{\Sigma}_\xi$ for some $p \in \underline{\Sigma}_\xi$. Since β is a Busemann function corresponding to a point $\xi \in \overline{\sigma}$ it follows that $\beta \circ \rho = \beta$. Hence ρ restricts to an isometry $\rho \colon \Sigma \to \underline{\Sigma}_\xi$ that fixes $K_p(\sigma_\xi)$. On the other hand it follows from the definition of ρ_ξ that $\rho_\xi \colon \Sigma \to \underline{\Sigma}_\xi$ is an isometry that fixes $K_p(\sigma_\xi)$ as well. Obviously there is just one such isometry and thus the retractions ρ and ρ_ξ coincide on every apartment $\Sigma \in \mathcal{A}_\xi^\sigma$. In view of Remark 8.8 this implies that ρ_ξ and ρ coincide on X_ξ. □

It will be convenient to denote the restriction of h to X_ξ by h_ξ.

Lemma 8.11. *The height function $h_\xi \colon X_\xi \to \mathbb{R}$ is invariant under ρ_ξ. In other words we have $h_\xi(\rho_\xi(x)) = h_\xi(x)$ for every $x \in X_\xi$.*

Proof. By definition h_ξ is the restriction of h to X_ξ and from Lemma 8.10 we know that ρ_ξ is the restriction of ρ to X_ξ. Therefore the assumption $h \in X^*_{\sigma,\underline{\Sigma}}$ gives us

$$h_\xi(\rho_\xi(x)) = h(\rho(x)) = h(x) = h_\xi(x)$$

for every $x \in X_\xi$. □

Note that $h_\xi \colon \underline{\Sigma}_\xi \to \mathbb{R}$ is the restriction of the linear function $h \colon \underline{\Sigma} \to \mathbb{R}$ and hence is linear itself. Thus Lemma 8.11 gives us the following.

Corollary 8.12. *The function h_ξ lies in $(X_\xi)^*_{\sigma_\xi,\underline{\Sigma}}$.*

In the following we will just write $X^*_\xi := (X_\xi)^*_{\sigma_\xi,\underline{\Sigma}}$.

8.2 REDUCTION OF THE HORIZONTAL DIMENSION

Recall that the initial goal of this section was to determine whether the system $(X_{h\geq r})_{r\in\mathbb{R}}$ is essentially k-connected for some given $k\in\mathbb{N}_0$. In this final section we reduce this question from the case where $\overline{\sigma}\subset\partial_\infty\underline{\Sigma}_{h\leq 0}$ to the special case where $\overline{\sigma}\cap\partial_\infty\underline{\Sigma}_{h=0}$ is empty. Without loss of generality we may therefore assume that $\overline{\sigma}\subset\partial_\infty\underline{\Sigma}_{h\leq 0}$ is non-empty.

Note that $\overline{\sigma}\cap\partial_\infty\underline{\Sigma}_{h=0}$ is the closure of a face σ_{hor} of σ. We refer to σ_{hor} as the horizontal face of σ with respect to h.

In order to describe the reduction process, we consider the Euclidean building X_ξ, the chamber $\sigma_\xi\subset\partial_\infty X_\xi$, and the height function $h_\xi\in X_\xi^*$ defined earlier in this chapter. Recall that σ_ξ was contained in $\partial_\infty((\underline{\Sigma}_\xi)_{h_\xi\leq 0})$. As above we can therefore observe that $\overline{\sigma_\xi}\cap\partial_\infty((\underline{\Sigma}_\xi)_{h\geq 0})$ is the closure of a face $\sigma_\xi^{\mathrm{hor}}$ of σ_ξ.

Lemma 8.13. *With the notation above we have*

$$\dim(\sigma_\xi^{\mathrm{hor}})=\dim(\sigma_{\mathrm{hor}})-1.$$

Proof. Let $\mathcal{B}=\{\alpha_i\colon 0\leq i\leq d\}$ be as in Definition 7.8 and let j be any index such that σ_{hor} is contained in $\partial_\infty(\ker(\alpha_j))$. The choice of ξ directly implies that ξ is a vertex of σ_{hor}. Note that the dimension of σ_{hor} is given by

$$\begin{aligned}\dim(\sigma_{\mathrm{hor}})&=\max_{1\leq i\leq d}(\dim(\ker(\alpha_i)\cap\underline{\Sigma}_{h=0}))-1\\&=\dim(\ker(\alpha_j)\cap\underline{\Sigma}_{h=0})-1.\end{aligned}$$

Further, we have a chain of inclusions

$$\ker(\alpha_i)\cap\underline{\Sigma}_{h=0}\cap\underline{\Sigma}_\xi\subseteq\ker(\alpha_j)\cap\underline{\Sigma}_{h=0}\cap\underline{\Sigma}_\xi\subsetneq\ker(\alpha_j)\cap\underline{\Sigma}_{h=0}\,. \tag{8.1}$$

for every $0\leq i\leq d$. Indeed, the first inclusion follows from the choice of j and the properness of the second inclusion follows from the observation that the ray $[\underline{v},\xi)$ is contained in $\ker(\alpha_j)\cap\underline{\Sigma}_{h=0}$ but not in $\underline{\Sigma}_\xi$. The fact that $\underline{\Sigma}_\xi$ is a hyperplane in $\underline{\Sigma}$ thus gives us

$$\dim(\ker(\alpha_j)\cap\underline{\Sigma}_{h=0}\cap\underline{\Sigma}_\xi)=\dim(\ker(\alpha_j)\cap\underline{\Sigma}_{h=0})-1. \tag{8.2}$$

Let α_i^ξ denote the restriction of α_i to $\underline{\Sigma}_\xi$. From Lemma 8.6 it follows that the walls bounding the sector $K_{\underline{v}}(\sigma_\xi)\subset\underline{\Sigma}_\xi$ can be described as $\ker(\alpha_i^\xi)=\ker(\alpha_i)\cap\underline{\Sigma}_\xi$ for $1\leq i\leq d$. Since the zero level of h_ξ in $\underline{\Sigma}_\xi$ is given by $\underline{\Sigma}_\xi\cap\underline{\Sigma}_{h=0}$ we get

$$\dim(\sigma_\xi^{\mathrm{hor}})=\max_{1\leq i\leq d}\dim(\ker(\alpha_i)\cap\underline{\Sigma}_\xi\cap\underline{\Sigma}_{h=0})-1.$$

By combining this equation with (8.1) and (8.2) we therefore obtain

$$\begin{aligned}\dim(\sigma_\xi^{\mathrm{hor}})&=\dim(\ker(\alpha_j)\cap\underline{\Sigma}_{h=0}\cap\underline{\Sigma}_\xi)-1\\&=\dim(\ker(\alpha_j)\cap\underline{\Sigma}_{h=0})-2\\&=\dim(\sigma_{\mathrm{hor}})-1.\end{aligned}$$

□

The next theorem summarizes what we have done so far in this chapter. It provides us one step of the promised reduction.

Theorem 8.14. *Let X be a Euclidean building, let $\Sigma \subset X$ be an apartment, let $\sigma \subset \partial_\infty \Sigma$ be a chamber, and let $v \in \Sigma$ be a special vertex. Let further $h \in X^*_{\sigma,v}$ be a height function such that $\sigma \subset \partial_\infty \Sigma_{h \leq 0}$. Suppose that the horizontal face σ_{hor} of σ with respect to h is non-empty.*

Then there is a vertex a Euclidean building Y of dimension $\dim(Y) - 1$, an apartment $\Pi \subset Y$, a chamber $\tau \subset \partial_\infty Y$, a special vertex $w \in \Pi$, and a height function $f \in Y^*_{\tau,w}$ with $\tau \subset \partial_\infty Y_{f \leq 0}$ such that the following properties are satisfied.

1. *For every $k \in \mathbb{N}_0$, the system $(X_{h \geq r})_{r \in \mathbb{R}}$ is essentially k-connected if and only if the system $(Y_{f \geq r})_{r \in \mathbb{R}}$ is essentially k-connected. The dimension of the horizontal face τ_{hor} of τ with respect to f is given by $\dim(\tau_{\text{hor}}) = \dim(\sigma_{\text{hor}}) - 1$.*

2. *For every cell $A \subset Y$ there is a cell $B \subset X$ such that $\operatorname{lk}_Y(A) \cong \operatorname{lk}_X(B)$. In particular we see that if $\widetilde{\Phi}$ is the Coxeter diagram of X and $\widetilde{\Psi}$ is the Coxeter diagram of Y then Ψ is a subdiagram of Φ. The boundary $\partial_\infty Y$ is isomorphic to the link of a special vertex in $\partial_\infty X$.*

3. *We have $\operatorname{th}(X_\xi) \geq \operatorname{th}(X)$. If X is locally finite then so is X_ξ.*

Proof. The first claim is given by Corollary 8.4 and the second claim is given Lemma 8.13. The claims in the third point are stated in Corollary 7.29 and Remark 7.30. The last claim is given in Remark 7.28. □

By iterating the construction of Theorem 8.14 we get the following corollary. It allows us to restrict ourselves to the case where σ_{hor} is empty.

Corollary 8.15. *Let X be a Euclidean building, let $\Sigma \subset X$ be an apartment, let $\sigma \subset \partial_\infty \Sigma$ be a chamber, and $v \in \Sigma$ be a special vertex. Let $h \in X^*_{\sigma,v}$ be a height function such that $\sigma \subset \partial_\infty \Sigma_{h \geq 0}$ and let $\tau = \sigma_{\text{hor}}$ denote the maximal horizontal face of σ with respect to h. There is a building X_τ of dimension $\dim(X) - \dim(\tau) - 1$, an apartment $\Sigma_\tau \subset X_\tau$, a chamber $\sigma_\tau \subset \partial_\infty \Sigma_\tau$, a special vertex $v_\tau \in \Sigma_\tau$, and a height function $h_\tau \in (X_\tau)^*_{\sigma_\tau, v_\tau}$ with $\sigma_\tau \subset \partial_\infty((\Sigma_\tau)_{h_\xi \leq 0})$ such that the following properties are satisfied.*

1. *For every $k \in \mathbb{N}_0$, the system $(X_{h \geq r})_{r \in \mathbb{R}}$ is essentially k-connected if and only if the system $((X_\tau)_{h_\tau \geq r})_{r \in \mathbb{R}}$ is essentially k-connected.*

2. *The horizontal face $(\sigma_\tau)_{\text{hor}}$ of σ_τ with respect to h_τ is empty.*

3. *If $\widetilde{\Phi}$ is the Coxeter diagram of X and $\widetilde{\Psi}$ is the Coxeter diagram of X_τ then Ψ is a subdiagram of Φ.*

4. *The thickness of X_τ satisfies $\operatorname{th}(X_\tau) \geq \operatorname{th}(X)$. If X is locally finite then so is X_τ.*

8.3 THE GEOMETRIC MAIN RESULT

We are now ready to prove the main theorem of this chapter. In order to formulate the result we have to recall that for every finite dimensional real vector space V we denote by $S(V)$ the space of positive homothety classes of non-trivial elements of V. In the following this notation will be used for vector spaces of the form $X^*_{\sigma,v}$ (see 4.3 for a definition).

Definition 8.16. Let X be a Euclidean building, let $\sigma \subset \partial_\infty \Sigma$ be a chamber, and let $v \in X$ be a special vertex. For each panel P of σ let $\alpha_P \in S(X^*_{\sigma,v})$ be the class of functions that are negative on $K_v(\sigma)$ and constant on $K_v(P)$. The set of all such classes will be denoted by

$$\mathcal{B}_v(\sigma) := \{\alpha_P \colon P \text{ is a panel of } \sigma\}.$$

Remark 8.17. Every system of representatives of $\mathcal{B}_v(\sigma)$ is a basis of $X^*_{\sigma,v}$. Thus $\mathcal{B}_v(\sigma)$ spans a simplex in $S(X^*_{\sigma,v})$.

In view of Remark 8.17 it makes sense to denote the convex hull of $\mathcal{B}_v(\sigma)$ in $S(X^*_{\sigma,v})$ by $\Delta_v(\sigma)$. In other words, $\Delta_v(\sigma)$ denotes the set of elements that are represented by non-trivial vectors of the form $\sum \lambda_P v_P$, where $\lambda_P \geq 0$ and $[v_P] \in \mathcal{B}_v(\sigma)$. For each $0 \leq k < \dim(V)$ it will be useful to consider the k-skeleton $\Delta_v(\sigma)^{(k)} \subset \Delta_v(\sigma)$. It can be described as the k-convex hull of $\mathcal{B}_v(\sigma)$ (see 2.49 for a definition), i.e. $\Delta_v(\sigma)^{(k)}$ is the set of elements that are represented by non-trivial vectors of the form $\sum\limits_{i=0}^{k} \lambda_i v_i$, where $\lambda_i \geq 0$ and $[v_i] \in \mathcal{B}_v(\sigma)$.

Theorem 8.18. *Let X be a d-dimensional Euclidean building, let Σ be an apartment in X, let $\sigma \subset \partial_\infty \Sigma$ be a chamber, let $v \in \Sigma$ be a special vertex, and let $h \in X^*_{\sigma,v}$. Assume that*

1. $\mathrm{Aut}(X)$ *acts strongly transitively on X,*
2. *X satisfies the* SOL*-property, and that*
3. *$\partial_\infty X$ satisfies the* SOL*-property.*

Then $(X_{h \geq r})_{r \in \mathbb{R}}$ is essentially contractible, i.e. essentially k-connected for every $k \in \mathbb{N}_0$, if and only if $[h]$ is not contained in $\Delta_v(\sigma)$.

If $[h]$ is contained in $\Delta_v(\sigma)$ and $0 \leq k < \dim(\Delta_v(\sigma))$ then $[h]$ is contained in $\Delta_v(\sigma)^{(k+1)} \setminus \Delta_v(\sigma)^{(k)}$ if and only if the system $(X_{h \geq r})_{r \in \mathbb{R}}$ is essentially k-connected but not essentially $(k+1)$-acyclic.

Proof. Suppose first that $[h]$ is not contained in $\Delta_v(\sigma)$, or equivalently, that $\sigma \not\subseteq \partial_\infty \Sigma_{h \leq 0}$. It suffices to show that every map $S^n \to X_{h \geq s}$ is null-homotopic. Thus let f be such a map. Let $s \in \mathbb{R}$ be a fixed number. Since $\sigma \not\subseteq \partial_\infty \Sigma_{h \leq 0}$ we see that there is some (interior) point $\eta \in \sigma$ such that the ray $[x, \eta)$ is contained in $X_{h \geq s}$ for every $x \in X_{h \geq s}$. Thus it follows that for all $x, y \in X_{h \geq s}$ there is a number $t \geq 0$

such that $[x,\eta)(t)$ is contained in $K_y(\sigma)$. Since S^n is compact, this implies that there is a sector $K_w(\sigma)$ and a constant $t \geq 0$ such that $[f(z),\eta)(t) \in K_w(\sigma) \cap X_{h\geq s}$ for every $z \in S^n$. Now the claim follows from the simple observation that $K_w(\sigma) \cap X_{h\geq s}$ is convex. Next we consider the case where $[h]$ is contained in $\Delta_v(\sigma)$, or equivalently, that $\sigma \subseteq \partial_\infty \Sigma_{h\leq r}$. Let $\tau = \sigma_{\mathrm{hor}}$ be the horizontal face of σ with respect to h.

Recall that the case $\tau = \emptyset$ was proven in Theorem 4.15 and Theorem 5.25. Let us therefore assume that τ is non-empty and let $k \in \mathbb{N}_0$ be such that $[h] \in \Delta_v(\sigma)^{(k+1)} \setminus \Delta_v(\sigma)^{(k)}$. Note that k is given by $k = \dim(X) - \dim(\tau) - 3$. In this case Corollary 8.15 equips us with a $(k+2)$-dimensional Euclidean building X_τ, a height function $h_\tau \in (X_\tau)^*_{\sigma_\tau,v_\tau}$ and a chamber $\sigma_\tau \subset \partial_\infty((\Sigma_\tau)_{h_\tau\leq 0})$ such that the horizontal part of σ_τ with respect to h_τ is empty. Furthermore Corollary 8.15 tells us for every $m \in \mathbb{N}_0$, that the system $(X_{h\geq r})_{r\in\mathbb{R}}$ is essentially m-connected (respectively m-acyclic) if and only if the system $((X_\tau)_{h_\tau\geq r})_{r\in\mathbb{R}}$ is essentially m-connected (respectively m-acyclic). In view of the third claim in Corollary 8.15 it follows from the sphericity of the links in X and $\partial_\infty X$ that the links in X_τ and $\partial_\infty X_\tau$ are spherical as well. This allows us to apply Theorem 4.15 and Theorem 5.25 a second time that tell us that $((X_\tau)_{h_\tau\geq r})_{r\in\mathbb{R}}$ is essentially $(\dim(X_\tau) - 2)$-connected but not essentially $(\dim(X_\tau) - 1)$-acyclic. Thus the claim follows from $\dim(X_\tau) = k + 2$. □

9

CHEVALLEY GROUPS, BOREL GROUPS, AND THEIR S-ARITHMETIC SUBGROUPS

In order to apply the topological results on buildings which we obtained so far to the computation of the Σ invariants of a group we need the group to act on a building. One source of groups acting on Euclidean buildings is the class of Chevalley groups over valued fields. To define these groups, we have to recall some classical facts from the theory of complex Lie algebras. All the details can be found in [21].

9.1 BACKGROUND ON LIE ALGEBRAS

For the rest of this chapter we fix a complex semisimple Lie algebra $\mathcal{L}$. Let κ be its Killing form and let $\mathcal{H} \subset \mathcal{L}$ be a fixed a Cartan subalgebra. Since the restriction of κ on $\mathcal{H}$ is non-degenerate by [21, Corollary 8.2.] we can identify $\mathcal{H}$ with its dual $\mathcal{H}^*$ via $H \mapsto \kappa(H, -)$. For each $\gamma \in \mathcal{H}^*$ let $H'_\gamma \in \mathcal{H}$ be the corresponding element with respect to this identification. The set of roots corresponding to $\mathcal{H}$ will be denoted by $\Phi \subset \mathcal{H}^*$. Let $\mathcal{V} = \langle \Phi \rangle_{\mathbb{R}}$ be its real span. We will view $\mathcal{V}$ as a Euclidean vector space where the inner product is given by

$$\kappa^*(\alpha, \beta) := \kappa(H'_\alpha, H'_\beta).$$

The Euclidean space $(\mathcal{V}, \kappa^*)$ is going to be the standard apartment of the Euclidean building we are going to describe in section 9.4. Let $\Delta := \{\alpha_1, \dots, \alpha_l\} \subset \Phi$ be a system of simple roots, let $\Phi^+ \subset \Phi$ be the corresponding set of positive roots, and let $\widetilde{\alpha} \in \Phi^+$ be the highest root. The Weyl group associated to Φ will be denoted by W_Φ, i.e. $W_\Phi < \mathrm{Isom}(\mathcal{V})$ is the group generated by the reflections s_α through the hyperplanes

$$W_\alpha := \{v \in \mathcal{V} \colon \kappa^*(\alpha, v) = 0\}.$$

For every two elements $v, w \in \mathcal{V}$ let

$$\langle v, w \rangle := 2\frac{\kappa^*(v, w)}{\kappa^*(v, v)}.$$

With this notation we can express reflections by

$$s_\alpha(v) = v - \langle v, \alpha \rangle \alpha.$$

Recall that $\mathcal{L}$ can be decomposed as a direct sum

$$\mathcal{L} = \mathcal{H} \oplus \bigoplus_{\alpha \in \Phi} \mathcal{L}_\alpha$$

where

$$\mathcal{L}_\alpha = \{X \in \mathcal{L} \colon [H, X] = \alpha(H)X, \ \forall H \in \mathcal{H}\}$$

for every root $\alpha \in \Phi$. It can be shown that each $\mathcal{L}_\alpha$ is one-dimensional and that $[\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}] \subset \mathcal{H}$. In order to state the following result (see [31, Theorem 1]), it will be useful to rescale the elements H'_α by defining

$$H_\alpha := \frac{2H'_\alpha}{\kappa(H'_\alpha, H'_\alpha)}$$

for each root α. In the case of simple roots we will write $H_i = H_{\alpha_i}$.

Theorem 9.1. *For $(H_i)_{i=1}^{l}$ as above there are elements $X_\alpha \in \mathcal{L}_\alpha$ for each $\alpha \in \Phi$ such that the set*

$$\{H_i \colon i = 1, \ldots, l\} \cup \{X_\alpha \colon \alpha \in \Phi\}$$

is a linear basis of $\mathcal{L}$ and the following relations are satisfied.

(a) $[H_i, H_j] = 0$.

(b) $[H_i, X_\alpha] = \alpha(H_i)X_\alpha$.

(c) $[X_\alpha, X_{-\alpha}] = H_\alpha$ *where H_α is an integral linear combination of the H_i.*

(d) $[X_\alpha, X_\beta] = \pm(r+1)X_{\alpha+\beta}$ *if $\alpha + \beta \in \Phi$ where*

$$r := \max\{k \in \mathbb{N}_0 \colon \beta - k\alpha \in \Phi\}.$$

(e) $[X_\alpha, X_\beta] = 0$ *if $\alpha + \beta \neq 0$ and $\alpha + \beta \notin \Phi$.*

In the following we will refer to

$$\{H_i \colon i = 1, \ldots, l\} \cup \{X_\alpha \colon \alpha \in \Phi\}$$

as the *Chevalley basis* of $\mathcal{L}$. In order to define Chevalley groups we have to consider representations of $\mathcal{L}$. Thus, for the rest of this chapter we fix an irreducible, faithful, finite dimensional representation

$$\rho \colon \mathcal{L} \to \operatorname{End}(V).$$

By [31, Theorem 3] there is a finite set $\Psi \subset \mathcal{H}^*$ of so-called *weights* such that V splits into a direct sum

$$V = \bigoplus_{\mu \in \Psi} V_\mu$$

of non-trivial weight spaces

$$V_\mu = \{v \in V \colon \rho(H)(v) = \mu(H)v, \ \forall H \in \mathcal{H}\}.$$

A non-zero vector $v \in V_\mu$ is called a weight vector with respect to μ. The following key observation leads to the definition of Chevalley groups (see [31, Corollaries 1 and 2]).

Theorem 9.2. *The $\mathcal{L}$-module V contains a lattice M, i.e. the $\mathbb{Z}$-span of a basis of V, such that the following hold.*

1. *M is invariant under the action of $\frac{\rho(X_\alpha)^n}{n!}$.*
2. *The lattice M splits as a direct sum $M = \bigoplus_{\mu \in \Psi} M_\mu$ with $M_\mu := M \cap V_\mu$.*
3. *The part of $\mathcal{L}$ that leaves M invariant decomposes as a direct sum*

$$\mathcal{L}_{\mathbb{Z}} = \mathcal{H}_\rho \oplus \bigoplus_{\alpha \in \Phi} \langle X_\alpha \rangle_{\mathbb{Z}}, \text{ where}$$

$$\mathcal{H}_\rho = \{H \in \mathcal{H} \colon \mu(H) \in \mathbb{Z}, \forall \mu \in \Psi\}.$$

9.2 CHEVALLEY GROUPS AND THEIR ASSOCIATED SUBGROUPS

Let $\mathcal{L}$, $\mathcal{H}$, Φ, $\mathcal{V}$, V, M, and Ψ be as above. In order to construct Chevalley groups there are three choices to make, two of which we already made. The first choice was the semisimple Lie algebra $\mathcal{L}$ which by [21, Theorem 14.2.] is the same as choosing the root system Φ. The second choice was the representation ρ of $\mathcal{L}$. The last choice we have to make is the choice of an arbitrary field K. All the other choices we made, i.e. the choice of the invariant lattice M, the choice of the Cartan subalgebra $\mathcal{H}$, as well as the choice of the system of simple roots Δ do not change the structure of the resulting Chevalley group up to isomorphism. The Chevalley group $\mathcal{G} := \mathcal{G}(\Phi, \rho, K)$ is defined by its action on the vector space $V^K := M \otimes_{\mathbb{Z}} K$. The next theorem helps us to analyze this action (see [31, Corollary 3]).

Theorem 9.3. *The vector space V^K decomposes as $V^K = \bigoplus_{\mu \in \Psi} V^K_\mu$ where $V^K_\mu := M_\mu \otimes_{\mathbb{Z}} K$.*

When K is clear from the context we will just write $V = V^K$. From the theory of real Lie groups it is well known that the exponential map gives us a function from the Lie algebra to the Lie group. A similar approach can be taken in the case of arbitrary fields. Here we have to restrict the exponentiation to elements of finite order. The following observation gives us examples of such elements.

Lemma 9.4. *For every $\alpha \in \Phi$ the element $\rho(X_\alpha) \in \mathrm{End}(V)$ is* nilpotent, *i.e. there is a number $n \in \mathbb{N}$ such that $\rho(X_\alpha)^n = 0$.*

Proof. The claim follows directly from [31, Lemma 11]. □

In view of Lemma 9.4 the following definition makes sense.

Definition 9.5. For every $t \in K$ let $x_\alpha(t) = \sum_{n \geq 0} \frac{t^n \cdot X_\alpha^n}{n!}$. Here X_α is identified with its image $\rho(X_\alpha) \in \mathrm{End}(V)$.

A quick computation shows that $x_\alpha(s)x_\alpha(t) = x_\alpha(s+t)$ for every two elements $s, t \in K$. Since $x_\alpha(0) = \mathrm{id}_V$ this shows in particular that $x_\alpha(t) \in \mathrm{GL}(V)$.

Definition 9.6. The *Chevalley group* $\mathcal{G} = \mathcal{G}(\Phi, \rho, K)$ is the subgroup of $\mathrm{GL}(V)$ generated by the elements $x_\alpha(t)$ for every $\alpha \in \Phi$ and every $t \in K$.

The following definition provides us with some families of elements in $\mathcal{G}$ which are of special importance.

Definition 9.7. For every $\alpha \in \Phi$ and every $t \in K^\times$ we define the elements $w_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t)$, $h_\alpha(t) = w_\alpha(t)w_\alpha(1)^{-1}$, and $\omega_\alpha = w_\alpha(1)$.

The following list of relations in $\mathcal{G}$ can be proved by analyzing the action of $\mathcal{G}$ on V (see the list below [31, Lemma 20]).

Proposition 9.8. *The following relations are satisfied.*

(a) $x_\alpha(s)x_\alpha(t) = x_\alpha(s+t)$.

(b) $[x_\alpha(s), x_\beta(t)] = \prod\limits_{i,j>0} x_{i\alpha+j\beta}(c_{i,j,\alpha,\beta}t^i s^j)$ *if* $\alpha + \beta \neq 0$ *and where the constants* $c_{i,j,\alpha,\beta} \in K$ *are suitably defined.*

(c) $\omega_\alpha h_\beta(t)\omega_\alpha^{-1} = \prod\limits_{\gamma\in\Phi} h_\gamma(t_\gamma)$ *for suitably defined elements* $t_\gamma \in K^\times$ *that do not depend on the representation space.*

(d) $\omega_\alpha x_\beta(t)\omega_\alpha^{-1} = x_{\omega_\alpha(\beta)}(ct)$ *where* $c \in \{-1, 1\}$.

(e) $h_\alpha(t)x_\beta(s)h_\alpha(t)^{-1} = x_\beta(t^{\langle\beta,\alpha\rangle}s)$.

Equipped with the elements defined in Definition 9.7 we can now define some subgroups of $\mathcal{G}$ which are of particular importance in the study of $\mathcal{G}$.

Definition 9.9. For every Chevalley group $\mathcal{G}(\Phi, \rho, K)$ let

1. $\mathcal{U}_\alpha$ be the group $\{x_\alpha(t) \colon t \in K\}$ for some $\alpha \in \Phi$,
2. $\mathcal{U}$ be the group generated by all subgroups $\mathcal{U}_\alpha$ with $\alpha \in \Phi^+$,
3. $\mathcal{T}$ be the group generated by all elements of the form $h_\alpha(t)$,
4. $\mathcal{B}$ be the group generated by $\mathcal{U}$ and $\mathcal{T}$, and
5. $\mathcal{N}$ be the group generated by all elements of the form $w_\alpha(t)$.

The groups $\mathcal{U}$,$\mathcal{T}$, and $\mathcal{B}$ are called the *unipotent*, *torus*, and *Borel subgroup* of $\mathcal{G}$.

Lemma 9.10. *The following relations between the groups* $\mathcal{U}$, $\mathcal{T}$, $\mathcal{B}$, *and* $\mathcal{N}$ *are satisfied.*

(a) *$\mathcal{U}$ is normal in $\mathcal{B}$ and $\mathcal{B} = \mathcal{T} \ltimes \mathcal{U}$.*

(b) *$\mathcal{T}$ is normal in $\mathcal{N}$. If K has more than 3 elements then $\mathcal{N}$ is the full normalizer of $\mathcal{T}$.*

(c) *The map $s_\alpha \mapsto w_\alpha(1)\mathcal{T}$, where α runs over Φ, extends to an isomorphism $W_\Phi \to \mathcal{N}/\mathcal{T}$.*

The following lemma describes the structure of $\mathcal{T}$ (see [31, Lemma 28]).

Lemma 9.11. *The function*

$$h\colon (K^\times)^l \to \mathcal{T},\ (t_i)_{i=1}^l \mapsto \prod_{i=1}^{l} h_i(t_i)$$

is an epimorphism.

9.3 FROM CHEVALLEY GROUPS TO RGD-SYSTEMS AND BN-PAIRS

In order to define the buildings on which $\mathcal{G}$ acts we have to recall what an RGD-system is. A comprehensive treatment of RGD- and VRGD-systems can be found in [2] and [33]. Before turning to Chevalley groups again, we define the general notion of an RGD-system for an arbitrary group G. We start by recalling the notion of an open interval between two roots.

Definition 9.12. Let (W, S) be a spherical Coxeter system and let Ω denote the set of roots in the Coxeter complex $\Sigma(W, S)$. Recall that in this combinatorial setting, a root is a halfspace in $\Sigma(W, S)$ bounded by a wall and is said to be positive if it contains the chamber corresponding to the identity. For every two roots $\alpha, \beta \in \Phi$ we define $[\alpha, \beta] \subset \Phi$ to be the set of roots γ such that $\alpha \cap \beta \subset \gamma$. Let further $(\alpha, \beta) := [\alpha, \beta] \backslash \{\alpha, \beta\}$. The sets $[\alpha, \beta]$ and (α, β) are called the *closed* and the *open interval* between α and β.

Remark 9.13. If the set of roots Ω in Definition 9.12 comes from a root system in some ambient Euclidean space, then the closed and open intervals of every two roots $\alpha, \beta \in \Omega$ are given by

$$[\alpha, \beta] = \{i\alpha + j\beta \in \Omega\colon i, j \geq 0\}$$

and

$$(\alpha, \beta) = \{i\alpha + j\beta \in \Omega\colon i, j > 0\}.$$

Notation 9.14. For each group G let G^* denote the set of non-trivial elements of G.

Definition 9.15. Let (W, S) be a spherical Coxeter system and let Ω denote the set of roots in the Coxeter complex $\Sigma(W, S)$. For each $s \in S$ let $\alpha_s \in \Omega$ denote the positive root corresponding to s. A triple

$(G, (U_\alpha)_{\alpha\in\Omega}, T)$ consisting of a group G and subgroups $T \leq G$ and $U_\alpha \leq G$ for every $\alpha \in \Omega$ is called an RGD-*system* of type (W, S) if the following hold.

(RGD 0) For all $\alpha \in \Omega$, $U_\alpha \neq \{1\}$.

(RGD 1) For all $\alpha, \beta \in \Omega$ with $\alpha \neq \pm\beta$, we have the inclusion $[U_\alpha, U_\beta] \leq \langle U_\gamma \colon \gamma \in (\alpha, \beta)\rangle$.

(RGD 2) For every $s \in S$ and every $u \in U^*_{\alpha_s}$ there is an element $m(u) \in U_{-\alpha_s} u U_{-\alpha_s}$ such that $m(u)U_\alpha m(u)^{-1} = U_{s(\alpha)}$ for all $\alpha \in \Omega$.

(RGD 3) For all $s \in S$, $U_{-\alpha_s} \not\leq \langle U_\alpha \colon \alpha \in \Omega^+\rangle =: U$.

(RGD 4) $G = T\langle U_\alpha \colon \alpha \in \Omega\rangle$.

(RGD 5) $T \leq \bigcap_{\alpha\in\Phi} N_G(U_\alpha)$.

We will denote the set of reflections corresponding to the simple roots by $S_\Delta = \{s_\alpha \colon \alpha \in \Delta\}$. The following theorem is discussed in [2, Section 7.9.2].

Theorem 9.16. *The triple* $(\mathcal{G}, (\mathcal{U}_\alpha)_{\alpha\in\Phi}, \mathcal{T})$ *is an* RGD-*system of type* (W_Φ, S_Δ). *The function m in* (RGD 2) *can be defined by*

$$m(x_\alpha(\lambda)) = w_{-\alpha}(-\lambda^{-1}).$$

It turns out that RGD-systems can by used to construct BN-pairs. Before we continue we quickly recall what that is (see [2, Definition 6.55]).

Definition 9.17. Let G be a group and let B, N be subgroups of G such that G is generated by B and N. The pair (B, N) is called a BN-*pair* of G if the intersection $T := B \cap N$ is normal in N, and the quotient $W := N/T$ admits a set of generators S such that the following conditions are satisfied.

(BN 1) For $s \in S$ and $w \in W$, $sBw \subset BswB \cup BwB$.

(BN 2) For $s \in S$, $sBs^{-1} \not\leq B$.

The tuple (G, B, N, S) is called a *Tits system*.

The next theorem describes the transition from RGD-systems to BN-pairs (see [2, Theorem 7.115.]).

Theorem 9.18. *Let* $(G, (U_\alpha)_{\alpha\in\Omega}, T)$ *be an* RGD-*system of type* (W, S). *Let U be the group generated by the groups U_α with $\alpha \in \Omega^+$. Let further $B = TU$ and $N = \langle T, \{m(u) \colon u \in U^*_{\alpha_s}, s \in S\}\rangle$ where $\alpha_s \in \Omega$ is the root corresponding to $s \in S$ which contains the identity element of W. Then the tuple (G, B, N, S) is a Tits system, $B \cap N = T$, and the map $\pi \colon N \to \mathrm{Sym}(\Omega)$ given by $nU_\alpha n^{-1} = U_{\pi(n)(\alpha)}$ induces an isomorphism $N/T \to W$ where W is identified with its image in* $\mathrm{Sym}(\Omega)$.

It is important to note that the definition of a BN-pair does not imply that N/T is a spherical Coxeter group. In fact in section 9.4 we will construct a BN-pair $(\widetilde{\mathcal{B}}, \mathcal{N})$ for $\mathcal{G}$ such that $\mathcal{N}/(\widetilde{\mathcal{B}} \cap \mathcal{N})$ is a Euclidean Coxeter group.

The following corollary summarizes what the notation in the previous sections suggests.

Corollary 9.19. *Let S_Δ be as above and let $\mathcal{G}$, $\mathcal{B}$, $\mathcal{N}$, and $\mathcal{T}$ be as in Definition 9.9. The tuple $(\mathcal{G}, \mathcal{B}, \mathcal{N}, S_\Delta)$ is a Tits system.*

Proof. In view of Theorem 9.18 and Theorem 9.16, this follows directly from the construction of the Chevalley group $\mathcal{G}$ and its subgroups $\mathcal{B}$, $\mathcal{N}$, and $\mathcal{T}$. □

9.4 FROM VALUED ROOT GROUP DATA TO BN-PAIRS

There is a second BN-pair associated to $\mathcal{G}$ in the case where the field K possesses a discrete valuation.

Definition 9.20. Let K be a field. A function $v \colon K \to \mathbb{Z} \cup \{\infty\}$ is called a *discrete valuation* if

1. $v(x \cdot y) = v(x) + v(y)$ for all $x, y \in K$,
2. $v(x + y) \geq \min\{v(x), v(y)\}$ for all $x, y \in K$, and
3. $v(x) = \infty \Leftrightarrow x = 0$.

For the rest of this section, we will assume that K has a fixed discrete valuation v. This valuation can be used to produce a valued RGD-system.

Definition 9.21. Let $(G, (U_\alpha)_{\alpha \in \Omega}, T)$ be an RGD-system. For each $\alpha \in \Omega$ let $\phi_\alpha \colon U_\alpha^* \to \mathbb{Z} \cup \{\infty\}$ be a function such that

$$\phi_\alpha(u) = \infty \Leftrightarrow u = id.$$

The tuple $(G, (U_\alpha)_{\alpha \in \Omega}, (\phi_\alpha)_{\alpha \in \Omega})$ is called a *valued* RGD-*system* if the following properties are satisfied.

(VRGD 0) Each ϕ_α is surjective.

(VRGD 1) For each $\alpha \in \Omega$ and each $k \in \mathbb{Z}$ the set

$$U_{\alpha,k} := \{u \in U_\alpha \colon \phi_\alpha(u) \geq k\}$$

is a group.

(VRGD 2) For all $\alpha, \beta \in \Omega$ with $\alpha \neq \pm\beta$ and all $k, l \in \mathbb{Z}$,

$$[U_{\alpha,k}, U_{\beta,l}] \subseteq \prod_{\gamma \in (\alpha,\beta)} U_{\gamma, p_\gamma k + q_\gamma l},$$

where p_γ and q_γ are suitably defined positive real numbers.

(VRGD 3) For all $\alpha, \beta \in \Omega$ and every two elements $u \in U_\alpha^*$, $x \in U_\beta^*$, the number

$$\phi_{s_\alpha(\beta)}(m(u)^{-1}xm(u)) - \phi_\beta(x)$$

is independent of x, where m is the function given in (RGD 2).

(VRGD 4) For each $\alpha \in \Omega$ and all $u \in U_\alpha^*$, $x \in U_\alpha^*$

$$\phi_{-\alpha}(m(u)^{-1}xm(u)) - \phi_\alpha(x) = -2\phi_\alpha(u).$$

In our case the functions ϕ_α can be defined explicitly as follows.

Definition 9.22. For every root $\alpha \in \Phi$ let $\phi_\alpha \colon \mathcal{U}_\alpha \to \mathbb{Z},\ x_\alpha(t) \mapsto v_p(t)$.

The following definition can be found in [3, Proposition 3.2.].

Proposition 9.23. *The triple* $(\mathcal{G}, (\mathcal{U}_\alpha)_{\alpha\in\Phi}, (\phi_\alpha)_{\alpha\in\Phi})$ *is a* VRGD*-system. The function m in* (RGD 2) *can be chosen as in Theorem 9.16, i.e.*

$$m(x_\alpha(t)) = w_{-\alpha}(-t^{-1}) \textit{ for every } \alpha \in \Phi \textit{ and every } t \in K^\times.$$

The general construction of the Euclidean BN-pair associated to a VRGD-system, as described in [33, Chapter 14], involves a choice of an identification of the abstract set of roots Ω with some explicit root system in some ambient Euclidean space. Since in our situation, the Coxeter system (W, S) comes from the Weyl group $W_\Phi \subset \mathrm{Isom}(\mathcal{V})$ together with a set S_Δ of generators there is no choice to make. Thus the following is a simplified version of the construction in [33, Chapter 14]. Before we can move on we have to associate the Euclidean space $(\mathcal{V}, \kappa^*)$ with the structure of a Euclidean Coxeter complex. This complex will serve as a metric model for the standard apartment in the Euclidean building associated to the VRGD-system defined above.

Definition 9.24. Let $\widetilde{\Phi} := \Phi \times \mathbb{Z}$. For each $(\alpha, k) \in \Phi \times \mathbb{Z}$, we define $s_{\alpha,k} \in \mathrm{Isom}(\mathcal{V})$ to be the reflection given by $s_{\alpha,k}(v) = s_\alpha(v) + k\alpha^V$ where $\alpha^V := \frac{2\alpha}{\kappa^*(\alpha,\alpha)}$ denotes the *coroot* of α, and let

$$H_{\alpha,k} = \{v \in \mathcal{V} \colon \kappa^*(v, \alpha) = k\}$$

denote its invariant hyperplane.

A quick calculation shows that the set of all hyperplanes $H_{\alpha,k}$ is invariant under the action of the reflections $s_{\alpha,k}$. Thus the space $\mathcal{V}$ can be endowed with the structure of a Euclidean Coxeter complex whose set of walls is given by $\{H_{\alpha,k} \colon (\alpha, k) \in \widetilde{\Phi}\}$.

Definition 9.25. Let $W_{\widetilde{\Phi}}$ be the group generated by all reflections $s_{\alpha,k}$.

By [2, Section 10.1.3.], the group $W_{\widetilde{\Phi}}$ is generated by the set

$$S_{\widetilde{\Delta}} := \{s_{\alpha,0} \colon \alpha \in \Delta\} \cup \{s_{\widetilde{\alpha},1}\}.$$

Note further that $W_{\widetilde{\Phi}}$ contains the Weyl group $W_\Phi = \langle \{s_{\alpha,0} \colon \alpha \in \Delta\} \rangle$ as a subgroup. In order to define the groups involved in the Euclidean BN-pair, we have to define an action of $\mathcal{N}$ on this complex (see [33, Proposition 14.4.]).

Proposition 9.26. *There is an epimorphism $\pi \colon \mathcal{N} \to W_{\widetilde{\Phi}}$ that satisfies $\pi(m(u)) = s_{\alpha,-\phi_\alpha(u)}$ for every $\alpha \in \Phi$ and every $u \in \mathcal{U}_\alpha^*$.*

Definition 9.27. Let $\widetilde{\mathcal{T}}$ denote the kernel of π. Let further $\widetilde{\mathcal{U}}$ be the group generated by $\bigcup_{\alpha \in \Phi^+} \mathcal{U}_{\alpha,0}$ and $\bigcup_{\alpha \in \Phi^-} \mathcal{U}_{\alpha,1}$ and let $\widetilde{\mathcal{B}} = \widetilde{\mathcal{T}}\widetilde{\mathcal{U}}$.

Definition 9.28. For each $\alpha_i \in \Delta$, let $m_i \in \mathcal{N}/\widetilde{\mathcal{T}}$ denote the element represented by $m_{\alpha_i}(x_{\alpha_i}(1))$ and let $S = \{m_i \colon 1 \le i \le l\}$. Let further $m_0 \in \mathcal{N}/\widetilde{\mathcal{T}}$ be the element represented by $m_{-\widetilde{\alpha}}(u)$ for some $u \in \mathcal{U}_{-\widetilde{\alpha}}$ with $\phi_{-\widetilde{\alpha}}(u) = 1$ and let $\widetilde{S} = \{m_i \colon 0 \le i \le l\}$.

Now we have all ingredients to define the Tits system which we will study in the next sections (see [33, Theorem 14.38.]).

Theorem 9.29. *The tuple $(\mathcal{G}, \widetilde{\mathcal{B}}, \mathcal{N}, \widetilde{S})$ is a Tits system. Furthermore we have $\widetilde{\mathcal{T}} = \widetilde{\mathcal{B}} \cap \mathcal{N}$ and the map $\widetilde{S} \to S_{\widetilde{\Delta}}$, defined by $m_i \mapsto s_{\alpha_i,0}$ for $1 \le i \le l$ and $m_0 \mapsto s_{\widetilde{\alpha},1}$, extends to an isomorphism $\mathcal{N}/\widetilde{\mathcal{T}} \cong W_{\widetilde{\Phi}}$.*

9.5 FROM BN-PAIRS TO BUILDINGS

In this section we recall how a BN-pair (B, N) gives rise to a simplicial building $\Delta(B, N)$. The simplices of $\Delta(B, N)$ are given by cosets of standard parabolic subgroups. Recall that for a Coxeter system (W, S) and a subset $J \subset S$ the subgroup of W generated by J is denoted by W_J. These groups give rise to the construction of standard parabolic subgroups of $\mathcal{G}$ (see [2, Proposition 6.27.]).

Proposition 9.30. *Let (G, B, N, S) be a Tits system and let $J \subset S$ be a subset. The union of double cosets $P_J := \bigcup_{w \in W_J} B\widetilde{w}B$ is a group where $\widetilde{w} \in N$ is any representative of w.*

The groups of the form P_J will be called *standard parabolic subgroups*.

Definition 9.31. Let (G, B, N, S) be a Tits system. Let $\Delta(B, N)$ be the poset of the cosets of standard parabolic subgroups gP_J as in Proposition 9.30, ordered by reverse inclusion. For every $s \in S$ let $P_{\hat{s}}$ be the parabolic subgroup corresponding to the set $S \backslash \{s\}$.

Remark 9.32. It can be shown (see [2, Theorem 6.43.]) that every group $B \le P \le G$ is already of the form P_J for some subset $J \subset S$. In particular, it follows that the poset $\Delta(B, N)$ does not depend on the choice of S.

The poset $\Delta(B,N)$ can be viewed as an abstract simplicial complex where the set of vertices is given by the cosets of maximal proper parabolic subgroups and the simplices are given by subsets of the form $\{gP_{\hat{s}} \colon s \in J\}$ for some subset $J \subset S$ and some element $g \in G$. Note that the cells of maximal dimension correspond to cosets of the form gB. Here, the standard parabolic group G is viewed as the empty face in $\Delta(B,N)$. The next theorem summarizes some properties of $\Delta(B,N)$. The following result can be easily derived from what we have seen so far (see e.g. [2, Exercise 6.54] and its solution starting on page 708).

Theorem 9.33. *Let (G,B,N,S) be a Tits system and let $W = N/N \cap B$ be the associated Coxeter group. The complex $\Delta(B,N)$ satisfies the following properties.*

1. *$\Delta(B,N)$ is a building.*
2. *The complex $\Sigma(B,N) := \{nP \colon n \in N, B \leq P < G\}$ is an apartment.*
3. *A system of apartments is given by $\mathcal{A}_{B,N} := \{g\Sigma(B,N) \colon g \in G\}$.*
4. *The action of G on $\Delta(B,N)$ by left multiplication is strongly transitive with respect to $\mathcal{A}_{B,N}$.*

9.6 A METRIC FOR THE BUILDINGS

In this section we equip the buildings $\Delta(\mathcal{B},\mathcal{N})$ and $\Delta(\widetilde{\mathcal{B}},\mathcal{N})$ with a canonical CAT(1)- respectively CAT(0)-metric. Furthermore, we will explain the geometric relationship between these buildings.

Definition 9.34. Let E be the chamber in $\mathcal{V}$ defined by

$$\{v \in \mathcal{V} \colon \kappa^*(v,\alpha_i) > 0 \text{ for every } 1 \leq i \leq l \text{ and } \kappa^*(v,\widetilde{\alpha}) < 1\}.$$

For every $1 \leq i \leq l$, let v_i be the vertex of E not lying in $H_{\alpha_i,0}$ and let $v_0 = 0$. Let further

$$\mathcal{K} = \{v \in \mathcal{V} \colon \kappa^*(v,\alpha_i) > 0 \text{ for every } 1 \leq i \leq l\}$$

be the cone corresponding to E and v_0.

Recall that the vertices of the complex $\Sigma(\widetilde{\mathcal{B}},\mathcal{N})$ are given by the set of cosets of the form nP_i where $n \in \mathcal{N}$ and $P_i := P_{\widehat{m_i}}$ is the maximal parabolic subgroup associated to some $m_i \in \widetilde{S}$.

Definition 9.35. Let $\Sigma(V, W_{\widetilde{\Phi}})$ denote the abstract simplicial complex that is given by the triangulation of $\mathcal{V}$ with the hyperplanes $H_{\alpha,k}$ and let $\iota_{\mathcal{V}} \colon |\Sigma(V, W_{\widetilde{\Phi}})| \to \mathcal{V}$ denote the canonical homeomorphism.

The following result is a translation of [2, Proposition 10.13.] to the present setting. It allows us to endow the abstract complex $\Sigma(\widetilde{\mathcal{B}},\mathcal{N})$ with the structure of a Euclidean vector space and thus to consider affine height functions on it.

Proposition 9.36. *The map*

$$f\colon \Sigma(\widetilde{\mathcal{B}},\mathcal{N})^{(0)} \to \Sigma(V,W_{\widetilde{\Phi}})^{(0)},\ nP_i \mapsto \pi(n)(v_i)$$

is an incidence-preserving $\mathcal{N}$-equivariant bijection between the vertex sets of the complexes $\Sigma(\widetilde{\mathcal{B}},\mathcal{N})$ and $\Sigma(V,W_{\widetilde{\Phi}})$. By composing $|f|$ with $\iota_{\mathcal{V}}$ we get an $\mathcal{N}$-equivariant homeomorphism $\iota\colon |\Sigma(\widetilde{\mathcal{B}},\mathcal{N})| \to \mathcal{V}$.

In view of Proposition 9.36, there is a canonical metric on $|\Sigma(\widetilde{\mathcal{B}},\mathcal{N})|$ given by the pullback of the metric on $\mathcal{V}$. Since for every apartment Σ in $\Delta(\widetilde{\mathcal{B}},\mathcal{N})$, there is an isomorphism $f_\Sigma\colon \Sigma \to \Sigma(\widetilde{\mathcal{B}},\mathcal{N})$, we can endow $|\Sigma|$ with a metric by pulling back the metric from $|\Sigma(\widetilde{\mathcal{B}},\mathcal{N})|$ to $|\Sigma|$ via $|f_\Sigma|$. Since every two points x, y lie in some common apartment, this gives us a function $d_{\Delta(\widetilde{\mathcal{B}},\mathcal{N})}\colon |\Delta(\widetilde{\mathcal{B}},\mathcal{N})| \times |\Delta(\widetilde{\mathcal{B}},\mathcal{N})| \to \mathbb{R}$. It can be shown (see [2, Theorem 11.16.]) that this function is a well defined CAT(0)-metric.

Remark 9.37. Note that under the above identification, the boundary $\sigma := \partial_\infty\mathcal{K}$ can be viewed as a chamber in $\partial_\infty(\Delta(\widetilde{\mathcal{B}},\mathcal{N}))$.

In the following we will always think of $\Delta(\widetilde{\mathcal{B}},\mathcal{N})$ as the geometric realization equipped with the metric described above. Note that the boundary $\partial_\infty\mathcal{V}$ can be endowed with the angular metric. Thus, by the same procedure as above, we can think of $\partial_\infty(\Delta(\widetilde{\mathcal{B}},\mathcal{N}))$ as a spherical building whose metric is given by pulling back the metric from the boundary of the standard apartment. The following theorem shows that the group $\mathcal{B}$ appears naturally in the study of the action of $\mathcal{G}$ on $\Delta(\widetilde{\mathcal{B}},\mathcal{N})$ (see [33, Theorem 14.46.]).

Theorem 9.38. *The group $\mathcal{B}$ is the stabilizer of σ in $\mathcal{G}$.*

Note that Theorem 9.38 in particular shows that the map

$$\lambda\colon \Delta(\mathcal{B},\mathcal{N}) \to \partial_\infty(\Delta(\widetilde{\mathcal{B}},\mathcal{N})),\ g\mathcal{B} \mapsto g(\sigma)$$

is a well defined embedding. The following result relates the image $\lambda(\Delta(\mathcal{B},\mathcal{N}))$ of this embedding to the complex

$$\partial_\infty(\Delta(\widetilde{\mathcal{B}},\mathcal{N}),\mathcal{A}_{\widetilde{\mathcal{B}},\mathcal{N}}) := \bigcup_{\Sigma\in\mathcal{A}_{\widetilde{\mathcal{B}},\mathcal{N}}} \partial_\infty|\Sigma|.$$

It will be useful for us to know that λ is an isomorphism (see [33, Theorem 14.47.])

Theorem 9.39. *With the notation above, we have an isomorphism*

$$\lambda\colon \Delta(\mathcal{B},\mathcal{N}) \to \partial_\infty(\Delta(\widetilde{\mathcal{B}},\mathcal{N}),\mathcal{A}_{\widetilde{\mathcal{B}},\mathcal{N}}).$$

Remark 9.40. Note that since $\partial_\infty(\Delta(\widetilde{\mathcal{B}},\mathcal{N}),\mathcal{A}_{\widetilde{\mathcal{B}},\mathcal{N}})$ inherits the metric from $\partial_\infty(\Delta(\widetilde{\mathcal{B}},\mathcal{N}))$, it follows that $\Delta(\mathcal{B},\mathcal{N})$ can be endowed with the metric given by pulling back the metric from $\partial_\infty(\Delta(\widetilde{\mathcal{B}},\mathcal{N}),\mathcal{A}_{\widetilde{\mathcal{B}},\mathcal{N}})$ via λ.

9.7 THE CASE OF p-ADIC VALUATIONS

In this section, we will translate the results of the last section to more concrete groups. We keep the definitions from the previous sections and consider the case where $K = \mathbb{Q}$. Thus from now on, $\mathcal{G}$ is a Chevalley group defined by generators of the form $x_\alpha(t)$ with $t \in \mathbb{Q}$. Let v_p denote the p-adic valuation on $\mathbb{Q}$ for some prime number $p \in \mathbb{N}$. As it is discussed in the previous section this leads to the construction of a spherical BN-pair $(\mathcal{B}, \mathcal{N})$ and a Euclidean BN-pair $(\widetilde{\mathcal{B}}_p, \mathcal{N})$. We will denote the corresponding embedding of the spherical building $\Delta(\mathcal{B}, \mathcal{N})$ in the boundary of the Euclidean building $X_p := \Delta(\widetilde{\mathcal{B}}_p, \mathcal{N})$ by X_p^∞. The standard apartments in these buildings will be denoted by $\Sigma_p^\infty = \Sigma(\mathcal{B}, \mathcal{N})$ and $\Sigma_p = \Sigma(\widetilde{\mathcal{B}}_p, \mathcal{N})$ respectively. In order to formulate our results we have to choose some embedding of $\mathcal{G}$ into the group of invertible matrices $\mathrm{GL}_d(\mathbb{Q})$. To do so, we fix a basis $\{b_1, \dots, b_d\}$ of the lattice M and identify $\mathcal{G}$ with the matrix representation corresponding to this basis. It will be important to us to view $\mathcal{G}$ and its subgroups as topological groups. Recall that the p-adic valuation on $\mathbb{Q}$ provides us with a p-adic absolute value given by $|p^k \frac{a}{b}|_p = p^{-k}$ where a and b are relatively prime to p and that the p-adic topology on $\mathbb{Q}$ is the topology induced by that absolute value. By associating $M_d(\mathbb{Q})$ with the product topology it follows easily that the subspace topology turns $\mathcal{G}$ and its subgroups into topological groups. The corresponding topological groups will be denoted by $\mathcal{G}_p$, $\mathcal{N}_p$, $\mathcal{U}_p$, etc. If there is no need to refer to p we will just drop the index. The groups we are interested in appear as groups of R-points in $\mathcal{G}$ where R is a subring of $\mathbb{Q}$ containing 1.

Definition 9.41. Let $R \leq \mathbb{Q}$ be a subring containing 1 and let $M_d(R)$ denote the set of $d \times d$ matrices with coefficients in R. For every subgroup $H \leq \mathcal{G}$ let $H(R) = H \cap M_d(R)$.

Remark 9.42. The set of R-points of a subgroup of $\mathcal{G}$ indeed forms a group. This follows from the basic rule $A^{-1} = \det(A)^{-1} A^{adj}$ relating the inverse of a matrix to its adjoint matrix and the fact that $\mathcal{G} \leq \mathrm{SL}_n(\mathbb{Q})$ which follows from [31, Lemma 11].

Definition 9.43. For every finite set of prime numbers $S \subset \mathbb{N}$ let A_S denote the subring of $\mathbb{Q}$ consisting of the elements $x \in \mathbb{Q}$ with $v_p(x) \geq 0$ for every $p \in S$. Let further $\mathcal{O}_S$ denote the subring of $\mathbb{Q}$ consisting of the elements $x \in \mathbb{Q}$ with $v_p(x) \geq 0$ for every prime p that is *not* contained in S. In the case where $S = \{p\}$ is a singleton we will just write A_p and $\mathcal{O}_p$.

Remark 9.44. Note that we could also write $\mathcal{O}_S = \mathbb{Z}[1/N]$ where $N = \prod_{p \in S} p$.

Lemma 9.45. *The set $M_d(A_p)$ is open in $M_d(\mathbb{Q})$ with respect to the p-adic topology.*

Proof. This follows directly from the observation that there is a positive number $\varepsilon > 0$ such that A_p is the open ball around 0 with radius $1+\varepsilon$. □

It will be very useful that many relations between the Chevalley group $\mathcal{G}$ and its subgroups still hold for their R-points if $R = A_S$ or $R = \mathcal{O}_S$.

Lemma 9.46. *Let $R \leq \mathbb{Q}$ be a subring of the form A_S or $\mathcal{O}_S$. The following properties of groups of R-points are satisfied.*

(*a*) *$\mathcal{G}(R)$ is generated by the set $\{x_\alpha(t) : \alpha \in \Phi,\ t \in R\}$.*

(*b*) *$\mathcal{B}(R) = \mathcal{U}(R)\mathcal{T}(R)$.*

(*c*) *$\mathcal{U}(R) = \{\prod_{\alpha\in\Phi^+} x_\alpha(t_\alpha) : t_\alpha \in R\}$.*

(*d*) *$\mathcal{T}(R) = \{\prod_{i=1}^{l} h_{\alpha_i}(t_{\alpha_i}) : t_{\alpha_i} \in R^\times\}$.*

Proof. The statements (b), (c) and (d) are covered by [31, Lemma 49]. Further by [31, Corollary 3] the statement (a) holds in the case where R is a Euclidean domain. Thus the claim follows from the fact that localizations of Euclidean domains are Euclidean (see [18, Theorem 3.33]). □

Lemma 9.47. *For every root $\alpha \in \Phi$ the element $t_{\alpha,k} := s_{-\alpha,k}s_\alpha \in W_{\widetilde{\Phi}}$ acts on $\mathcal{V}$ by $t_{\alpha,k}(v) = v - k\alpha^V$.*

Proof. Recall that the action of s_α can be written down explicitly by $s_\alpha(v) = v - \kappa^*(v,\alpha)\alpha^V$ and that $s_{\alpha,k}$ is given by $s_{\alpha,k}(v) = s_\alpha(v) + k\alpha^V$. Thus

$$s_{-\alpha,k}(s_\alpha(v)) = s_{-\alpha}(s_\alpha(v)) - k\alpha^V = v - k\alpha^V.$$

□

Recall that $\widetilde{\mathcal{T}}_p$ was defined to be the kernel of the map $\pi\colon \mathcal{N}_p \to W_{\widetilde{\Phi}}$ induced by $\pi(m(x_\alpha(t))) = s_{\alpha,-v_p(t)}$ for every $\alpha \in \Phi$ and every $t \in \mathbb{Q}$. The following lemma provides us with an explicit description of $\widetilde{\mathcal{T}}_p$.

Lemma 9.48. *With the notation above we have $\widetilde{\mathcal{T}}_p = \{\prod_{i=1}^{l} h_i(t_i) : t_i \in A_p^\times\}$. Further we have $\pi(h_i(t_i)) = t_{\alpha_i,k_i}$ where $k_i = v_p(t_i)$.*

Proof. By definition the group $\widetilde{\mathcal{T}}_p$ fixes $\mathcal{V}$ pointwise. In particular $\widetilde{\mathcal{T}}_p$ fixes $\partial_\infty\mathcal{V}$ pointwise and therefore $\widetilde{\mathcal{T}}_p \subset \mathcal{B} \cap \mathcal{N} = \mathcal{T}$. Thus by Lemma 9.11 it follows that every element of $\widetilde{\mathcal{T}}_p$ can be written as a product $\prod_{i=1}^{l} h_i(t_i)$ for some elements $t_i \in \mathbb{Q}^\times$. The image of such a generator $h_i(t_i)$ under π is given by

$$\begin{aligned}
\pi(h_i(t_i)) &= \pi(w_{\alpha_i}(t_i)w_{\alpha_i}(1)^{-1}) \\
&= \pi(w_{\alpha_i}(t_i))\pi(w_{\alpha_i}(1)^{-1}) \\
&= \pi(m_{-\alpha_i}(-t_i^{-1}))\pi(m_{-\alpha_i}(-1)) \\
&= s_{-\alpha_i,-v_p(-t_i^{-1})}s_{-\alpha_i,-v_p(-1)} \\
&= s_{-\alpha_i,v_p(t_i)}s_{-\alpha_i,0} \\
&= s_{-\alpha_i,v_p(t_i)}s_{\alpha_i}.
\end{aligned}$$

Thus by defining $k_i = v_p(t_i)$ Lemma 9.47 implies $\pi(h_i(t_i)) = t_{\alpha_i,k_i}$. Suppose that $w = \prod_{i=1}^{l} h_i(t_i)$ lies in $\widetilde{\mathcal{T}}_p$, i.e. $\pi(w) = \mathrm{id} \in W_{\widetilde{\Phi}}$. From the above equality it follows that

$$\begin{aligned}
w(v) \quad &= \pi(\prod_{i=1}^{l} h_i(t_i))(v) \quad &&= \prod_{i=1}^{l} \pi(h_i(t_i))(v) \\
&= \prod_{i=1}^{l} t_{\alpha_i,k_i}(v) &&= \prod_{i=1}^{l}(v - v_p(t_i)\alpha_i^V).
\end{aligned}$$

Since $\Delta = \{\alpha_i\colon 1 \le i \le l\}$ is a basis of $\mathcal{V}$ the condition $v = w(v)$ implies $v_p(t_i) = 0$ for every $1 \le i \le l$. Now the claim follows from the simple fact that an element $t \in \mathbb{Q}$ is a unit in $A_p^\times$ if and only if $v_p(t) = 0$. □

The following result will be crucial to approximate the canonical action of $\prod_{p \in S} \mathcal{U}_p$ on the building $\prod_{p \in S} X_p$ by the diagonal action of $\mathcal{U}(\mathcal{O}_S)$.

Theorem 9.49. *Let $S \subset \mathbb{N}$ be a finite set of primes. The diagonal embedding*

$$\mathcal{U}(\mathcal{O}_S) \to \prod_{p \in S} \mathcal{U}_p$$

has dense image.

Proof. This is a direct consequence of the proof of [31, Theorem 20] where one has to replace $\mathcal{G}$ by $\mathcal{U}$. □

Lemma 9.50. *The intersection $\widetilde{\mathcal{B}}_p \cap \mathcal{U}$ coincides with $\mathcal{U}(A_p)$.*

Proof. Recall from Definition 9.27 that $\widetilde{\mathcal{B}}_p = \widetilde{\mathcal{T}}_p\widetilde{\mathcal{U}}_p$ where $\widetilde{\mathcal{U}}_p$ is the group generated by the two sets $\{x_\alpha(t)\colon \alpha \in \Phi^+,\ t \in A_p\}$ and $\{x_\alpha(t)\colon \alpha \in \Phi^-,\ t \in pA_p\}$ and where $\widetilde{\mathcal{T}}_p = \{\prod_{i=1}^{l} h_i(t_i)\colon t_i \in A_p^\times\}$ by Lemma 9.48. It thus follows from Lemma 9.46(c) that $\mathcal{U}(A_p)$ is contained in $\widetilde{\mathcal{B}}_p \cap \mathcal{U}_p$. On the other hand the construction gives us $\widetilde{\mathcal{T}}_p, \widetilde{\mathcal{U}}_p \subset \mathcal{G}(A_p)$ which shows that $\widetilde{\mathcal{B}}_p \cap \mathcal{U} = \mathcal{U}(A_p)$. □

10 Σ-INVARIANTS OF S-ARITHMETIC BOREL GROUPS

For the rest of this chapter let $\mathcal{G} = \mathcal{G}(\Phi, \rho, \mathbb{Q})$ be a Chevalley group, let $\mathcal{B} \subset \mathcal{G}$ be a Borel subgroup, and let $\Gamma = \mathcal{B}(\mathcal{O}_S)$ for some fixed finite set of prime numbers $S \subset \mathbb{N}$. Our first goal will be to prove that the action of $\mathcal{G}$ the product $X_S := \prod_{p \in S} X_p$ satisfies the conditions of Theorem 2.47 which allows us to compute the Σ-invariants of Γ by studying certain systems of superlevelsets in X_S.

10.1 FINITENESS PROPERTIES OF THE STABILIZERS

In this section we will determine the finiteness properties of the stabilizers of the action of Γ on X_S.

Lemma 10.1. *The group* $\bigcap_{p \in S} \widetilde{\mathcal{B}}_p \cap \Gamma$ *is of type* F_∞.

Proof. By construction we have $\widetilde{\mathcal{B}}_p = \widetilde{\mathcal{T}}_p \widetilde{\mathcal{U}}_p \leq \mathrm{SL}_d(A_p)$ and $\Gamma = \mathcal{B}(\mathcal{O}_S)$. Thus the intersection of these groups is contained in $\mathcal{B}(\bigcap_{p \in S} A_p \cap \mathcal{O}_S)$. Note that $\bigcap_{p \in S} A_p \cap \mathcal{O}_S = \mathbb{Z}$ and thus $\bigcap_{p \in S} \widetilde{\mathcal{B}}_p \cap \Gamma \leq \mathcal{B}(\mathbb{Z}) = \mathcal{T}(\mathbb{Z})\mathcal{U}(\mathbb{Z})$ by Lemma 9.46. Recall from Proposition 9.8 (a) and (b) that $\mathcal{U}(\mathbb{Z})$ is finitely generated and nilpotent. From Lemma 9.46 if further follows that $\mathcal{T}(\mathbb{Z}) = \{\prod_{i=1}^{l} h_{\alpha_i}(t_{\alpha_i}) \colon t_{\alpha_i} \in \{\pm 1\}\}$ and hence is finite. In particular we see that $\bigcap_{p \in S} \widetilde{\mathcal{B}}_p \cap \Gamma$ is a subgroup of a finitely generated virtually nilpotent group and thus is a finitely generated, virtually nilpotent group itself. Since the property F_∞ is invariant under commensurability, which follows for example from [4, Corollary 9], it remains to see that finitely generated nilpotent groups are of type F_∞. This follows from [19, 7.2. Exercise 1.] and the fact that finitely generated abelian groups are of type F_∞. □

Proposition 10.2. *Let A be a non-empty cell in X_S. The stabilizer* $\mathrm{St}_\Gamma(A)$ *is of type* F_∞.

Proof. Since X_S is locally finite it follows that all cell stabilizers are commensurable. Since being of type F_n is invariant under commensurability (see for example [4, Corollary 9]) it is sufficient to show that the stabilizer of some non-empty cell is of type F_∞. Recall that the stabilizer of the fundamental chamber in X_p with respect to the action of $\mathcal{G}$ is given by $\widetilde{\mathcal{B}}_p$. Thus the corresponding stabilizer of the diagonal action of Γ on X_S is given by $\bigcap_{p \in S} \widetilde{\mathcal{B}}_p \cap \Gamma$ which is of type F_∞ by 10.1. $\square$

10.2 COCOMPACTNESS OF THE ACTION

In this section we will show that the diagonal action of $\Gamma = \mathcal{B}(\mathcal{O}_S)$ on the product $X_S = \prod_{p \in S} X_p$ is cocompact. From the construction we already know that the action of $\mathcal{G}$ on X_p is cocompact for every $p \in S$. Our first goal is to show that this is still the case if we restrict the action to $\mathcal{B}$. Recall that in Proposition 9.36 we have identified the standard apartment $\Sigma_p \leq X_p$ with the Euclidean vector space $\mathcal{V}$ which was triangulated by the hyperplane arrangement $\{H_{\alpha,k} \colon \alpha \in \Phi, k \in \mathbb{Z}\}$. In particular this allows us to speak of the origin o_p of Σ_p. Let further

$$E_p = \{x \in \Sigma_p \colon \kappa^*(\alpha, x) \geq 0 \ \forall \alpha \in \Delta\}$$

denote the closed standard chamber in Σ_p.

Lemma 10.3. *The group $\mathcal{G}(A_p)$ is the stabilizer of the origin $o_p \in \Sigma_p$.*

Proof. We start by proving the inclusion $\mathcal{G}(A_p) \subset \mathrm{St}_{\mathcal{G}}(o_p)$. Lemma 9.46 tells us that $\mathcal{G}(A_p)$ is generated by all elements of the form $x_\alpha(t)$ with $\alpha \in \Phi$ and $t \in A_p$. Thus it suffices to show that these generators fix o_p. To see this recall that by construction $\widetilde{\mathcal{B}}_p$ is the pointwise stabilizer of E_p and in particular fixes o_p. Thus the generators $x_\alpha(t)$ of $\widetilde{\mathcal{B}}_p$ with $\alpha \in \Phi^+$ and $t \in A_p$ are contained in the stabilizer of o_p. On the other hand Proposition 9.26 tells us that o_p is fixed by ω_α for every $\alpha \in \Phi$. From these two types of generators we obtain from Proposition 9.8 that

$$\omega_\alpha x_\beta(t) \omega_\alpha^{-1} = x_{s_\alpha(\beta)}(\pm t)$$

stabilizes o_p for every $\alpha \in \Phi^+$ and every $t \in A_p$. Thus the inclusion $\mathcal{G}(A_p) \subset \mathrm{St}_{\mathcal{G}}(o_p)$ follows from the observation that every root $\alpha \in \Phi$ can be mapped into Φ^+ by an application of the reflection $s_{\widetilde{\alpha}}$. To see the reverse inclusion recall that $\mathcal{G}(A_p)$ is a standard parabolic subgroup by Remark 9.32. On the other hand we have just shown that $\mathcal{G}(A_p)$ contains the maximal parabolic subgroup P_0 which by Proposition 9.36 is the stabilizer of o_p. Thus the claim follows since obviously $\mathcal{G}(A_p) \neq \mathcal{G}$. $\square$

Proposition 10.4. *Let A be a cell in X_p. The stabilizer $\mathrm{St}_{\mathcal{G}}(A)$ is open in $\mathcal{G}_p$.*

Proof. From Lemma 9.45 and Lemma 10.3 it follows that the stabilizer of the origin o_p is open. Let $v \in \Sigma_p$ be a vertex of the same type as o_p such that the geodesic $[o_p, v]$ contains an interior point of E_p. Since $\mathcal{G}$ acts transitively on the set of vertices in X_p of a given type it follows that $\mathrm{st}_{\mathcal{G}}(v)$ is a conjugate of $\mathcal{G}(A_p)$ and hence open. Thus the intersection $\mathcal{G}(A_p) \cap \mathrm{st}_{\mathcal{G}}(v)$ is an open subgroup of $\mathcal{G}$ and fixes E_p pointwise. Since $\mathcal{G}$ is a topological group it follows that every subgroup of $\mathcal{G}$ containing $\mathcal{G}(A_p) \cap \mathrm{st}_{\mathcal{G}}(v)$ is open. Now the claim follows from the chamber transitivity of the action of $\mathcal{G}$ on X_p. $\square$

Corollary 10.5. *The stabilizers of the action of $\prod_{p \in S} \mathcal{U}_p$ on X_S are open in $\prod_{p \in S} \mathcal{U}_p$.*

Proof. For every $p \in S$ let C_p be a cell in X_p. From Proposition 10.4 we know that the stabilizer $\mathrm{St}_{\mathcal{G}_p}(C_p)$ is open in $\mathcal{G}_p$ and thus

$$\mathrm{St}_{\mathcal{U}_p}(C_p) = \mathcal{U}_p \cap \mathrm{St}_{\mathcal{G}_p}(C_p)$$

is also open. Since every cell in X_S can be written as $\prod_{p \in S} C_p$ we see that the stabilizer of C in $\prod_{p \in S} \mathcal{U}_p$ is a product of open subgroups and hence open. $\square$

The following theorem is crucial for showing that the action of Γ on X_S is cocompact (see [31, Theorem 17]).

Theorem 10.6. *Let $g \in \mathcal{G}$ be an arbitrary element. For every prime p there are elements $b_p \in \mathcal{B}$ and $g_p \in \mathcal{G}(A_p)$ such that $g = b_p g_p$.*

Proposition 10.7. *For every chamber $E \subset X_p$ there is an element $b \in \mathcal{B}$ such that $b(E)$ is contained in the star $\mathrm{st}_{\Sigma_p}(o_p)$ of o_p in Σ_p. In particular the action of $\mathcal{B}$ on X_p is cocompact.*

Proof. We start by considering the spherical building $\mathrm{lk}_{X_p}(o_p)$ and its apartment $\mathrm{lk}_{\Sigma_p}(o_p)$. For every root $\alpha \in \Phi^+$ let $\mathcal{U}_\alpha^{o_p}$ be the corresponding root group in $\mathrm{Aut}(\mathrm{lk}_{X_p}(o_p))$ and let $\mathcal{U}^{o_p}$ denote the group generated by the groups $\mathcal{U}_\alpha^{o_p}$ with $\alpha \in \Phi^+$. By [33, Proposition 18.17] it follows that for every root $\alpha \in \Phi^+$ and every element $u \in \mathcal{U}_\alpha^{o_p}$ there is an element $\tilde{u} \in \mathcal{U}_{\alpha,0} \leq \mathcal{U}$ such that $\tilde{u}_{|\,\mathrm{lk}_{X_p}(o_p)} = u$. On the other hand [2, Lemma 7.9] tells us that for every apartment $A \subset \mathrm{lk}_{X_p}(o_p)$ that contains the chamber in $\mathrm{lk}(o_p)$ that corresponds to E_p, there is an element $u \in \mathcal{U}_\alpha^{o_p}$ such that $u(A) = \mathrm{lk}_{\Sigma_p}(o_p)$. Thus we see that for every chamber $F \subset \mathrm{st}_{X_p}(o_p)$ there is a chamber $G \subset \mathrm{lk}_{\Sigma_p}(o_p)$ such that $F = b(G)$ for some appropriate choice of $b \in \mathcal{B}$. By the construction of X_p every chamber can be written as $g(E_p)$ for some $g \in \mathcal{G}$. By Theorem 10.6 we further have $g = bk$ for some appropriate elements $b \in \mathcal{B}$ and $k \in \mathcal{G}(A_p)$. Since k stabilizes o_p by Lemma 10.3 it follows that $k(E_p)$ is contained in $\mathrm{st}_{X_p}(o_p)$ and thus, by the observation above, can be written as $k(E_p) = b'(G)$ for some chamber $G \subset \mathrm{lk}_{\Sigma_p}(o_p)$ and some element $b' \in \mathcal{B}$. Thus $g(E_p) = bb'(G)$ which proves the claim. $\square$

Corollary 10.8. *The orbit of Σ_p under the action of $\mathcal{U}_p$ covers X_p, i.e.*

$$X_p = \bigcup_{\gamma \in \mathcal{U}_p} \gamma \cdot \Sigma_p.$$

Proof. Let $E \subset X_p$ be a chamber. By Proposition 10.7 there is a $b \in \mathcal{B}$ such that $b(E) \subset \mathrm{st}_{\Sigma_p}(o_p)$. Recall that $\mathcal{B} = \mathcal{T}\mathcal{U}$. Thus $b = tu$ for some appropriate choice of $t \in \mathcal{T}_p$ and $u \in \mathcal{U}$. Since $\mathcal{T}$ stabilizes Σ_p it follows that

$$u(E) = t^{-1}tu(E) = tb(E) \subset \Sigma_p$$

which proves the claim. □

Our goal is to extend this result to the case of the S-arithmetic part of the Borel group. In order to do so we will approximate the action of $\prod_{p \in S} \mathcal{U}$ on X_S by the diagonal action of $\mathcal{U}(\mathcal{O}_S)$. The next result makes more clear what we mean if we speak of approximating the action of a group.

Lemma 10.9. *Let G be a topological group acting on a cell complex X. Suppose that the stabilizers of cells are open in G. If $H \leq G$ is a dense subgroup of G then for every $g \in G$ and every cell A in X there is an element $h \in H$ such that $g(A) = h(A)$.*

Proof. Let $g \in G$ be an arbitrary element. Let $(h_n)_{n \in \mathbb{N}}$ be a sequence in H converging to g. We claim that there is an element h_n such that $h_n(A) = g(A)$. By assumption the stabilizer of A is open. Since G is a topological group it follows that $g\,\mathrm{St}_G(A)$ is an open neighborhood of g. Thus there is an n such that $h_n \in g\,\mathrm{St}_G(A)$ and the claim follows. □

Lemma 10.10. *The building X_S is covered by the orbit of $\Sigma_S = \prod_{p \in S} \Sigma_p$ under the action of $\mathcal{U}(\mathcal{O}_S)$.*

Proof. From Corollary 10.8 it follows that the orbit of Σ_S under the action of $\prod_{p \in S} \mathcal{U}_p$ covers X_S. On the other hand we know from Corollary 10.5 that the cell stabilizers of the action of $\prod_{p \in S} \mathcal{U}_p$ on X_S are open. In view of Lemma 10.9 the claim follows since the diagonal embedding

$$\mathcal{U}(\mathcal{O}_S) \to \prod_{p \in S} \mathcal{U}_p$$

has dense image by Theorem 9.49. □

Note that Lemma 10.10 in particular implies that the orbit of Σ_S under the action of $\Gamma = \mathcal{B}(\mathcal{O}_S) = \mathcal{T}(\mathcal{O}_S)\mathcal{U}(\mathcal{O}_S)$ covers X_S. To see that this action is also cocompact we consider the action of $\mathcal{T}(\mathcal{O}_S)$ on Σ_S. We start with the following easy observation.

Corollary 10.11. *The diagonal action of $\mathcal{T}(\mathcal{O}_S)$ on Σ_S is cocompact.*

Proof. By Lemma 9.48 we have $\pi(h_i(t_i)) = t_{\alpha_i,k_i}$ where $k_i = v_p(t_i)$ and $t_{\alpha_i,k_i}(v) = v - k_i\alpha_i^V$. Thus we see that the subgroup

$$\{h_i(p^z) \colon 1 \leq i \leq l, z \in \mathbb{Z}\} < \mathcal{T}(\mathcal{O}_p)$$

acts cocompactly on Σ_p and that $\mathcal{T}(\mathcal{A}_p)$ fixes Σ_p pointwise. Thus the multiplicativity of the functions h_i implies that the diagonal action of $\mathcal{T}(\mathcal{O}_S)$ on Σ_S is cocompact. □

Theorem 10.12. *The diagonal action of* Γ *on* X_S *is cocompact.*

Proof. From Lemma 9.46 we know that $\Gamma = \mathcal{T}(\mathcal{O}_S)\mathcal{U}(\mathcal{O}_S)$. Hence the claim follows from Lemma 10.10 and Corollary 10.11. □

10.3 THE STRUCTURE OF THE CHARACTER SPHERE

We continue our study of the group $\Gamma = \mathcal{B}(\mathcal{O}_S)$.

Definition 10.13. Let $\delta\colon \Gamma \to \mathcal{T}(\mathcal{O}_S)$ denote the canonical projection arising from the splitting $\Gamma = \mathcal{T}(\mathcal{O}_S)\mathcal{U}(\mathcal{O}_S)$.

Note that the kernel of δ is precisely $\mathcal{U}(\mathcal{O}_S)$. In order to understand the structure of the character sphere $S(\Gamma)$ of Γ it will be important to understand the relationship between $\mathcal{U}(\mathcal{O}_S)$ and the commutator subgroup of Γ.

Lemma 10.14. *The commutator subgroup* $[\Gamma, \Gamma]$ *lies in* $\mathcal{U}(\mathcal{O}_S)$. *The quotient* $Q := \mathcal{U}(\mathcal{O}_S)/[\Gamma, \Gamma]$ *is a torsion group.*

Proof. By Lemma 9.46 Γ is the semidirect product $\mathcal{T}(\mathcal{O}_S)\mathcal{U}(\mathcal{O}_S)$ where $\mathcal{T}(\mathcal{O}_S)$ is an abelian group. Thus we get the inclusion $[\Gamma, \Gamma] \subset \mathcal{U}(\mathcal{O}_S)$. For the second claim we recall relation (e) in Proposition 9.8 which tells us that

$$h_\alpha(t)x_\beta(s)h_\alpha(t)^{-1} = x_\beta(t^{\langle \beta, \alpha \rangle} s)$$

for $s \in \mathbb{Q}$ and $t \in \mathbb{Q}^*$. In the case $\beta = \alpha \in \Phi^+$ we thus get

$$h_\alpha(t)x_\alpha(s)h_\alpha(t)^{-1} = x_\alpha(t^2 s).$$

For a prime $p \in S$ we thus obtain

$$\begin{aligned} [h_\alpha(p), x_\alpha(s)] &= h_\alpha(p)x_\alpha(s)h_\alpha(p)^{-1}x_\alpha(s)^{-1} &&= x_\alpha(p^2 s)x_\alpha(s)^{-1} \\ &= x_\alpha(p^2 s - s) &&= x_\alpha(s)^{(p^2-1)}. \end{aligned}$$

Thus $x_\alpha(s)^{(p^2-1)} \in [\Gamma, \Gamma]$ for every $\alpha \in \Phi^+$ and every $s \in \mathcal{O}_S$. Let $q := p^2 - 1$. Now let $u \in \mathcal{U}(\mathcal{O}_S)$ be an arbitrary element. By Lemma 9.46 we can write

$$u = \prod_{i=1}^{n} x_{\beta_i}(t_i)$$

for some $\beta_i \in \Phi^+$ and $t_i \in \mathcal{O}_S$. Let $\overline{u}$ be the image of u in Q. Since Q is abelian, relation (a) in Proposition 9.8 gives us

$$\overline{u^q} = \overline{u}^q = \prod_{i=1}^{n} \overline{x_{\beta_i}(t_i)}^q = \prod_{i=1}^{n} \overline{x_{\beta_i}(t_i^q)}.$$

Now the claim follows since we have just seen that $\overline{x_{\beta_i}(t_i^q)} \in [\Gamma, \Gamma]$. □

Proposition 10.15. *The inclusion* $i\colon \mathcal{T}(\mathcal{O}_S) \to \Gamma$ *induces an isomorphism*

$$i^*\colon \operatorname{Hom}(\Gamma, \mathbb{R}) \to \operatorname{Hom}(\mathcal{T}(\mathcal{O}_S), \mathbb{R}), \chi \mapsto \chi \circ i.$$

The inverse isomorphism is given by

$$\delta^*\colon \operatorname{Hom}(\mathcal{T}(\mathcal{O}_S), \mathbb{R}) \to \operatorname{Hom}(\Gamma, \mathbb{R}), \chi \mapsto \chi \circ \tau.$$

Proof. Let $\gamma \in \Gamma$ be an arbitrary element. By Lemma 9.46 we can write $\gamma = tu$ with $t \in \mathcal{T}(\mathcal{O}_S)$ and $u \in \mathcal{U}(\mathcal{O}_S)$. By Lemma 10.14 we know that a power of u lies in $[\Gamma, \Gamma]$. Thus $\chi(u) = 0$ since $\mathbb{R}$ is torsion-free. Hence the restriction of χ to $\mathcal{T}(\mathcal{O}_S)$ can be taken to be the map $\overline{\chi}$ in the proposition. □

In view of Proposition 10.15 it suffices to consider the characters $\mathcal{T}(\mathcal{O}_S) \to \mathbb{R}$ in order to understand the characters of Γ. We want to split a character $\mathcal{T}(\mathcal{O}_S) \to \mathbb{R}$ into a product of basis elements.

Definition 10.16. For every $\alpha \in \Delta$ and every $p \in S$ let $\mathcal{T}_{\alpha,p}$ denote the subgroup of $\mathcal{T}(\mathcal{O}_S)$ consisting of elements of the form $h_\alpha(p^n)$ with $n \in \mathbb{Z}$. Let further $\mathcal{T}_S$ denote the group generated by the groups $\mathcal{T}_{\alpha,p}$.

The following lemma describes the structure of $\mathcal{T}_S$.

Lemma 10.17. *The canonical map*

$$f\colon \bigoplus_{p\in S}\bigoplus_{\alpha\in\Delta} \mathbb{Z} \to \mathcal{T}_S, (n_{\alpha,p})_{(p,\alpha)\in S\times\Delta} \mapsto \prod_{(p,\alpha)\in S\times\Delta} h_\alpha(p^{n_{\alpha,p}})$$

is an isomorphism. In particular $\mathcal{T}_S$ *canonically decomposes as* $\bigoplus_{p\in S}\bigoplus_{\alpha\in\Delta} \mathcal{T}_{\alpha,p}$.

Proof. It suffices to show that $\ker(f)$ is trivial. Thus suppose that

$$\gamma := \prod_{(p,\alpha)\in S\times\Delta} h_\alpha(p^{n_{\alpha,p}}) \in \ker(f).$$

Note that in particular γ acts trivially on Σ_p for every $p \in S$. Recall from 9.48 and Lemma 9.47 that the action of γ on $\Sigma_p \cong \mathcal{V}$ is given by

$$\begin{aligned} \gamma(x) &= \prod_{\alpha\in\Delta} \pi(h_\alpha(p^{n_{\alpha,p}}))(x) \\ &= \prod_{\alpha\in\Delta} t_{\alpha,n_{\alpha,p}}(x) \\ &= x - \sum_{\alpha\in\Delta} n_{\alpha,p}\alpha^V \end{aligned}$$

and hence $n_{\alpha,p} = 0$ for every $\alpha \in \Delta$ since Δ is a basis of $\mathcal{V}$. Now the claim follows since $p \in S$ was chosen arbitrarily. □

Lemma 10.18. *Every element in* $\mathcal{T}(\mathcal{O}_S)$ *can be uniquely written as* $t_\varepsilon t_S$ *where* t_ε *has finite order and* $t_S \in \mathcal{T}_S$.

Proof. From Lemma 9.46 we know that every element $\gamma \in \mathcal{T}(\mathcal{O}_S)$ can be written as $\gamma = \prod_{\alpha \in \Delta} h_\alpha(t_\alpha)$ with $t_\alpha \in \mathcal{O}_S^\times$. Thus $t_\alpha = \varepsilon_\alpha \prod_{p \in S} p^{k_\alpha}$ for some appropriate $k_\alpha \in \mathbb{Z}$ and $\varepsilon_\alpha \in \{\pm 1\}$. This gives us

$$\gamma = \prod_{\alpha \in \Delta} h_\alpha(t_\alpha) = \prod_{\alpha \in \Delta} (h_\alpha(\varepsilon_\alpha) \prod_{p \in S} h_\alpha(p^{k_{p,\alpha}})).$$

The predicted decomposition now follows by setting

$$t_\varepsilon := \prod_{\alpha \in \Delta} h_\alpha(\varepsilon_\alpha) \text{ and } t_S := \prod_{\alpha \in \Delta} \prod_{p \in S} h_\alpha(p^{k_{p,\alpha}}).$$

For the uniqueness we observe that if $\gamma = t_\varepsilon t_S = t'_\varepsilon t'_S$ has two such decompositions then $t_S'^{-1} t_S = t'_\varepsilon t_\varepsilon^{-1}$ has finite order. On the other hand Lemma 10.17 tells us that $\mathcal{T}_S$ is torsion-free and therefore we have $t_S = t'_S$ and hence $t_\varepsilon = t'_\varepsilon$. $\square$

Corollary 10.19. *The inclusion $\iota \colon \mathcal{T}_S \to \Gamma$ induces an isomorphism*

$$\iota^* \colon \operatorname{Hom}(\Gamma, \mathbb{R}) \to \operatorname{Hom}(\mathcal{T}_S, \mathbb{R}).$$

Proof. From Lemma 10.18 it follows that the inclusion $\mathcal{T}_S \to \mathcal{T}(\mathcal{O}_S)$ induces an isomorphism

$$\operatorname{Hom}(\mathcal{T}(\mathcal{O}_S), \mathbb{R}) \to \operatorname{Hom}(\mathcal{T}_S, \mathbb{R}).$$

Now the claim follows from Proposition 10.15. $\square$

In view of Corollary 10.19 we can now define the following characters of Γ.

Definition 10.20. For every $\alpha \in \Delta$ and $p \in S$ let $\chi_{\alpha,p} \colon \Gamma \to \mathbb{R}$ denote the unique extension of the character $\mathcal{T}_S \to \mathbb{R}$ that is induced by $h_\beta(t) \mapsto \langle \beta, \alpha \rangle v_p(t)$ for every $\alpha \in \Delta$. Let further

$$B_{\mathcal{G},\mathcal{B}}(S) = \{\chi_{\alpha,p} \colon \alpha \in \Delta, p \in S\}$$

denote the union of these characters.

The following proposition summarizes the observations above.

Remark 10.21. The set $B := B_{\mathcal{G},\mathcal{B}}(S)$ is a basis of $\operatorname{Hom}(\Gamma, \mathbb{R})$.

Proof. This follows from the fact that κ^* is non-degenerate and that Δ is a basis of $\mathcal{V}$. $\square$

10.4 EXTENDING CHARACTERS TO HEIGHT FUNCTIONS

In this section we will construct equivariant height functions for the action of Γ on X_S . I.e. for a given character $\chi \colon \Gamma \to \mathbb{R}$ we will define a continuous function $h \colon X_S \to \mathbb{R}$ such that the diagram

$$\begin{tikzcd} X_S \arrow[r, "h"] \arrow[d, "\gamma"'] & \mathbb{R} \arrow[d, "t_{\chi(\gamma)}"] \\ X_S \arrow[r, "h"] & \mathbb{R} \end{tikzcd}$$

commutes for every $\gamma \in \Gamma$. Here we denote by $t_{\chi(\gamma)} \colon \mathbb{R} \to \mathbb{R}$ the translation given by $x \mapsto x + \chi(\gamma)$. In order to formulate the second restriction that we are going to impose on the height functions h we have to consider the chamber $\sigma \subset \partial_\infty X_S$. Recall from Remark 9.37 that $\sigma \subset \partial_\infty \mathcal{V}$ denotes the chamber at infinity which is given as the boundary at infinity of the sector

$$\mathcal{K} = \{v \in \mathcal{V} \colon \kappa^*(v, \alpha) > 0 \text{ for every } \alpha \in \Delta\}$$

in $\mathcal{V}$. For each $p \in S$ let K_p denote the corresponding sector in Σ_p and let $\sigma_p := \partial_\infty K_p \subset \partial_\infty \Sigma_p$ be its boundary chamber at infinity. Finally let $\sigma_S \subset \partial_\infty \Sigma_S$ denote the boundary chamber at infinity of the sector $K_S := \prod\limits_{p \in S} K_p \subset \Sigma_S$. In the following we want h to be invariant under the retraction $\rho = \rho_{\Sigma_S, \sigma}$.

Definition 10.22. For each $p \in S$ let $\mathrm{pr}_p \colon \Sigma_S \to \Sigma_p$ be the canonical projection. For each $\alpha \in \Delta$ let further $\kappa_{\alpha,p} \colon \Sigma_p \to \mathbb{R}, v \mapsto \kappa^*(\alpha, \iota_p(v))$ be the linear form associated to α via κ^*. By composing these two functions with the retraction $\rho_{\Sigma_S,\sigma_S} \colon X_S \to \Sigma_S$ and inverting the sign we obtain the height functions

$$\mathrm{ht}_{\alpha,p} := -\kappa_{\alpha,p} \circ \mathrm{pr}_p \circ \rho_{\Sigma_S,\sigma_S} \colon X_S \to \mathbb{R}.$$

It follows directly from the definition that the functions $\mathrm{ht}_{\alpha,p}$ are ρ_{Σ_S,σ_S}-invariant, i.e. $\mathrm{ht}_{\alpha,p} \circ \rho_{\Sigma_S,\sigma_S} = \mathrm{ht}_{\alpha,p}$, and that the restriction of each $\mathrm{ht}_{\alpha,p}$ to Σ_S is a linear function. Thus we obtain $\mathrm{ht}_{\alpha,p} \in X_S^*$ where X_S^* denotes the real vector space of ρ_{Σ_S,σ_S}-invariant extensions of the linear forms in $\Sigma_S^* = \mathrm{Hom}(\Sigma_S, \mathbb{R})$ (see Definition 4.3). Since Δ is a basis of $\mathcal{V}$ it further follows that $B^{ht}_{\mathcal{G},\mathcal{B}}(S) := \{\mathrm{ht}_{\alpha,p} \colon \alpha \in \Delta,\ p \in S\}$ is a basis of X_S^*.

Definition 10.23. Let $\mathrm{ht} \colon \mathrm{Hom}(\Gamma, \mathbb{R}) \to X_S^*,\ \chi \mapsto \mathrm{ht}_\chi$ denote the isomorphism that extends the bijection

$$B_{\mathcal{G},\mathcal{B}}(S) \to B^{ht}_{\mathcal{G},\mathcal{B}}(S),\ \chi_{\alpha,p} \mapsto \mathrm{ht}_{\alpha,p}\,.$$

In order to describe the functions $\mathrm{ht}_{\alpha,p}$ more precisely we will identify the apartments Σ_p, where $p \in S$, with $\mathcal{V}$ via the equivariant homeomorphism $\iota_p \colon \Sigma_p \to \mathcal{V}$ from Proposition 9.36.

Remark 10.24. The diagonal action of an element $\prod\limits_{\alpha \in \Delta} h_\alpha(t_\alpha)$ on Σ_S is given by

$$x \mapsto x - \sum_{\alpha \in \Delta} \sum_{p \in S} v_p(t_\alpha) \iota_p^{-1}(\alpha^V).$$

Proof. This follows directly from Lemma 9.47 and Lemma 9.48. □

We are now ready to prove the equivariance in a specific situation.

Lemma 10.25. *Let $\chi \in \operatorname{Hom}(\Gamma, \mathbb{R})$ be a character. For every element $\gamma \in \mathcal{T}(\mathcal{O}_S)$ and every point $x \in \Sigma_S$ we have*

$$\operatorname{ht}_\chi(\gamma(x)) = \operatorname{ht}_\chi(x) + \chi(\gamma).$$

Proof. By the linearity of ht it suffices to proof the statement for the basis characters. Thus let $\chi = \chi_{\alpha,p}$ for some $\alpha \in \Delta$ and some $p \in S$ and let $\gamma = \prod_{\beta\in\Delta} h_\beta(t_\beta) \in \mathcal{T}(\mathcal{O}_S)$ be an arbitrary element. For every element $x \in \Sigma_S$ we have

$$\begin{aligned}
&\operatorname{ht}_\chi(\gamma.x)\\
&= \operatorname{ht}_{\alpha,p}((\prod_{\beta\in\Delta} h_\beta(t_\beta)).x)\\
&= \operatorname{ht}_{\alpha,p}(x - \sum_{\beta\in\Delta}\sum_{q\in S} v_q(t_\beta)\iota_q^{-1}(\beta^V))\\
&= \operatorname{ht}_{\alpha,p}(x) - \sum_{\beta\in\Delta}\sum_{q\in S} v_q(t_\beta)\operatorname{ht}_{\alpha,p}(\iota_q^{-1}(\beta^V))\\
&= \operatorname{ht}_{\alpha,p}(x) + \sum_{\beta\in\Delta}\sum_{q\in S} v_q(t_\beta)\kappa_{\alpha,p}\circ \operatorname{pr}_p \circ \rho_{\Sigma_S,\sigma_S}(\iota_q^{-1}(\beta^V))\\
&= \operatorname{ht}_{\alpha,p}(x) + \sum_{\beta\in\Delta}\sum_{q\in S} v_q(t_\beta)\kappa_{\alpha,p}\circ \operatorname{pr}_p(\iota_q^{-1}(\beta^V))\\
&= \operatorname{ht}_{\alpha,p}(x) + \sum_{\beta\in\Delta} v_p(t_\beta)\kappa_{\alpha,p}(\iota_p^{-1}(\beta^V))\\
&= \operatorname{ht}_{\alpha,p}(x) + \sum_{\beta\in\Delta} v_p(t_\beta)\kappa^*(\alpha,\beta^V)\\
&= \operatorname{ht}_{\alpha,p}(x) + \sum_{\beta\in\Delta} v_p(t_\beta)\langle\beta,\alpha\rangle\\
&= \operatorname{ht}_{\alpha,p}(x) + \sum_{\beta\in\Delta} \chi_{\alpha,p}(h_\beta(t_\beta))\\
&= \operatorname{ht}_{\alpha,p}(x) + \chi_{\alpha,p}(\prod_{\beta\in\Delta} h_\beta(t_\beta))\\
&= \operatorname{ht}_\chi(x) + \chi(\gamma)
\end{aligned}$$

□

The following lemma summarizes how characters and their height functions behave under the maps ρ and δ.

Lemma 10.26. *Let $\chi \in \operatorname{Hom}(\Gamma, \mathbb{R})$ be a character. Let further $\gamma \in \Gamma$ be an arbitrary element and let $\gamma = t_\gamma u_\gamma$ be the decomposition of γ into its torus part $t_\gamma \in \mathcal{T}(\mathcal{O}_S)$ and its unipotent part $u_\gamma \in \mathcal{U}(\mathcal{O}_S)$. For every $x \in X_S$ we have* *(a)* $\mathrm{ht}_\chi(\rho(x)) = \mathrm{ht}_\chi(x)$,

(b) $\chi(\delta(\gamma)) = \chi(\gamma)$,

(c) $\rho(u_\gamma(x)) = \rho(x)$,

(d) $\rho(t_\gamma(x)) = t_\gamma(\rho(x))$, *and*

(e) $\rho(\gamma(x)) = \delta(\gamma)(\rho(x))$.

Proof. The properties (a) and (b) follow directly from the construction of the basis characters and their corresponding height functions. To prove (c) and (d) let $x \in X$ be an arbitrary point and let Σ' be an apartment such that $x \in \Sigma'$ and $\sigma \subset \partial_\infty \Sigma'$. Let $\mathcal{K}_1$ be a common sector of Σ_S and Σ' such that $\partial_\infty \mathcal{K}_1 = \sigma$. From the construction of $\mathcal{U}(\mathcal{O}_S)$ we know that there is a subsector $\mathcal{K}_2$ of $\mathcal{K}_1$ that is fixed pointwise by u_γ. Let further $\Sigma'' = u_\gamma(\Sigma')$. Note that the restrictions of ρ to Σ' and Σ'' fix $\mathcal{K}_2$. Thus the isomorphisms $\rho \circ u_\gamma \colon \Sigma' \to \Sigma_S$ and $\rho \colon \Sigma' \to \Sigma_S$ fix $\mathcal{K}_2$ and hence coincide. In particular we obtain $\rho(u_\gamma(x)) = \rho(x)$ and hence (c). By the same argument we see that the isomorphisms $\rho \circ t_\gamma \colon \Sigma' \to \Sigma_S$ and $t_\gamma \circ \rho \colon \Sigma' \to \Sigma_S$ coincide since they coincide on a subsector $\mathcal{K}_2$ of $\mathcal{K}_1$ which proves (d). By applying the above rules we can derive (e).

$$\begin{aligned} \rho(\gamma(x)) \qquad & = \rho(t_\gamma u_\gamma(x)) && = \rho(t_\gamma(u_\gamma(x))) \\ = t_\gamma(\rho(u_\gamma(x))) & = t_\gamma(\rho(x)) && = \delta(\gamma)(\rho(x)) \qquad \square \end{aligned}$$

We are now ready to prove the desired equivariance of the height functions h_χ.

Corollary 10.27. *Let $\chi \colon \Gamma \to \mathbb{R}$ be a character. With the notation above we have*

$$h_\chi(\gamma(x)) = \chi(\gamma) + h_\chi(x) \text{ for every } x \in X_S \text{ and every } \gamma \in \Gamma.$$

Proof. In view of Lemma 10.25 and the properties in Lemma 10.26 we have

$$\begin{aligned} h_\chi(\gamma(x)) & = h_\chi(\rho(\gamma(x))) && = h_\chi(\delta(\gamma)(\rho(x))) \\ & = h_\chi(\rho(x)) + \chi(\delta(\gamma)) && = h_\chi(x) + \chi(\gamma). \qquad \square \end{aligned}$$

10.5 SIGMA INVARIANTS OF S-ARITHMETIC BOREL GROUPS

In this section we will prove the main results of this paper. In order to state these results we start by recalling and introducing some terminology. Let $\mathcal{G} = \mathcal{G}(\Phi, \rho, \mathbb{Q})$ be a Chevalley group, let $\mathcal{B} \subset \mathcal{G}$ be a Borel subgroup, and let $\Gamma = \mathcal{B}(\mathcal{O}_S)$ for some finite set of prime numbers $S \subset \mathbb{N}$. Let further $\Delta \subset \Phi$ be the set of simple roots that corresponds to $\mathcal{B}$. We consider the action of $\mathcal{G}$ on its corresponding Bruhat-Tits building $X_S = \prod_{p \in S} X_p$ that was described in Section 10.1. Recall from Definition 10.20 and Remark 10.21 that

$$B_{\mathcal{G},\mathcal{B}}(S) = \{\chi_{\alpha,p} \colon \alpha \in \Delta, p \in S\}$$

is a basis of $\mathrm{Hom}(\Gamma, \mathbb{R})$ and that

$$B^{ht}_{\mathcal{G},\mathcal{B}}(S) = \{\mathrm{ht}_{\alpha,p} \colon \alpha \in \Delta,\ p \in S\}$$

is a basis of X_S^*. Thus we see that the subset $\Delta_{\mathcal{G},\mathcal{B}}(S) \subset S(\Gamma)$ that is represented by the positive cone of $B_{\mathcal{G},\mathcal{B}}(S)$ has the structure of a closed simplex whose set of vertices is represented by $B_{\mathcal{G},\mathcal{B}}(S)$.

We recall the following well-known fact about the thickness of X_p. It follows for example from an application of the orbit-stabilizer theorem to the BN-characterization of panels in X_p.

Remark 10.28. The thickness of X_p is given by $p+1$.

We are now ready to prove our main result.

Theorem 10.29. *Let $\mathcal{G} = \mathcal{G}(\Phi, \rho, \mathbb{Q})$ be a Chevalley group, let $\mathcal{B} \subset \mathcal{G}$ be a Borel subgroup, and let $\Gamma = \mathcal{B}(\mathcal{O}_S)$ for some finite set of prime numbers $S \subset \mathbb{N}$. Suppose that*

1. *Φ is of type A_{n+1}, C_{n+1}, or D_{n+1} and that*
2. *every prime factor $p \in S$ satisfies $p \geq 2^n$ in the A_{n+1}-case, respectively $p \geq 2^{2n+1}$ in the other two cases.*

Then the Σ-invariants of Γ are given by

$$\Sigma^k(\Gamma) = S(\Gamma) \backslash \Delta_{\mathcal{G},\mathcal{B}}(S)^{(k)} \text{ for every } k \in \mathbb{N}.$$

Proof. Let $\Delta \subset \Phi$ be the set of simple roots that corresponds to $\mathcal{B}$. Let $\chi \colon \Gamma \to \mathbb{R}$ be a character and let $\mathrm{ht}_\chi \colon X_S \to \mathbb{R}$ be its corresponding height function as in Definition 10.23. From Corollary 10.27 we know that ht_χ is an equivariant extension of χ. Since the action of Γ on X_S is cocompact by Theorem 10.12 and the stabilizers of cells are of type F_∞ by Proposition 10.2, we can apply Theorem 2.47. This theorem tells us for every $k \in \mathbb{N}_0$ that $\chi \in \Sigma^{k+1}(\Gamma)$ if and only if $((X_S)_{\mathrm{ht}_\chi \geq r})_{r \in \mathbb{R}}$ is essentially k-connected. In order to apply Theorem 8.18, our geometric main result, we have to check that X_S and $\partial_\infty X_S$ satisfy the SOL-property and that $\mathrm{Aut}(X_S)$ acts strongly transitively on X_S. The second claim follows from the BN-characterization of X_S (see Theorem 9.33). To get the first claim we note that Remark 10.28 implies that the thickness of $\mathrm{th}(X_S)$ is given by $\mathrm{th}(X_S) = \min_{p \in S} p + 1$. Thus every link L in X_S satisfies $\mathrm{th}(L) \geq 2^n + 1$ in the A_{n+1} case, respectively $\mathrm{th}(L) \geq 2^{2n+1} + 1$ in the other two cases. This is exactly what we need to apply Theorem 2.39 which tells us that L satisfies the SOL-property in the case where L is irreducible. The SOL-property of the reducible links follows from the sphericity formula of joins (Lemma 2.21) applied to the restriction of the join decomposition of spherical buildings ([29, Proposition 1.15]) to the opposite complex of a chamber in L. Note that we also may apply Theorem 2.39 to $\partial_\infty X_S$ since the boundary at infinity of a thick building always satisfies $\mathrm{th}(\partial_\infty X_S) = \infty$. Before we can apply Theorem 8.18 we have to recall from Lemma 10.26, that ht_χ is ρ_{Σ_S,σ_S}-invariant, where Σ_S and $\sigma_S \subset \partial_\infty \Sigma_S$ are as in Section 10.4. Let $\mathcal{B}(\sigma_S) \subset S((X_S)^*)$ denote the set of classes α_P of functions that are negative on $K_0(\sigma_S)$ and constant on $K_0(P)$ for some panel P of σ_S and the origin $0 \in \Sigma_S$. The convex hull of $\mathcal{B}(\sigma_S)$ in $S(X_S^*)$ will be denoted by $\Delta(\sigma_S)$. An application of Theorem 8.18 to the present setting now gives us the following.

1. $[\mathrm{ht}_\chi] \notin \Delta(\sigma_S)$ if and only if $((X_S)_{\mathrm{ht}_\chi \geq r})_{r\in\mathbb{R}}$ is essentially contractible.

2. If $[\mathrm{ht}_\chi] \in \Delta(\sigma_S)$ and $0 \leq k < \dim(\Delta(\sigma_S))$ then $[h]$ is contained in $\Delta(\sigma_S)^{(k+1)} \backslash \Delta(\sigma_S)^{(k)}$ if and only if $(X_{\mathrm{ht}_\chi \geq r})_{r\in\mathbb{R}}$ is essentially k-connected but not essentially $(k+1)$-acyclic.

Note that the construction of the functions $\mathrm{ht}_{\alpha,p}$ (Definition 10.22) implies that their union $B^{ht}_{\mathcal{G},\mathcal{B}}(S)$ is a system of representatives of $\mathcal{B}(\sigma_S)$. Since ht_χ is the image of the isomorphism $\mathrm{ht}\colon \mathrm{Hom}(\Gamma,\mathbb{R}) \to X_S^*$ which extends the bijection

$$B_{\mathcal{G},\mathcal{B}}(S) \to B^{ht}_{\mathcal{G},\mathcal{B}}(S),\ \chi_{\alpha,p} \mapsto \mathrm{ht}_{\alpha,p}$$

we see that $[\mathrm{ht}_\chi] \in \Delta(\sigma_S)^{(k)}$ if and only if $[\chi] \in \Delta_{\mathcal{G},\mathcal{B}}(S)^{(k)}$ which proves the claim. □

In view of Theorem 2.48 we can translate Theorem 10.29 to the following result about finiteness properties of subgroups of $\mathcal{B}(\mathcal{O}_S)$.

Corollary 10.30. *Let $\mathcal{G} = \mathcal{G}(\Phi,\rho,\mathbb{Q})$ be a Chevalley group, let $\mathcal{B} \subset \mathcal{G}$ be a Borel subgroup, and let $\Gamma = \mathcal{B}(\mathcal{O}_S)$ for some finite set of prime numbers $S \subset \mathbb{N}$. Suppose that*

1. *Φ is of type A_{n+1}, C_{n+1}, or D_{n+1} and that*

2. *every prime factor $p \in S$ satisfies $p \geq 2^n$ in the A_{n+1}-case, respectively $p \geq 2^{2n+1}$ in the other two cases.*

Then for every subgroup $[\Gamma,\Gamma] \leq H \leq \Gamma$ and every $k \in \mathbb{N}$ we have

H is of type F_k if and only if $H \nsubseteq \ker(\chi)$ for every $\chi \in \Delta_{\mathcal{G},\mathcal{B}}(S)^{(k)}$.

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